

# Modular data: the algebraic combinatorics of conformal field theory

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**Abstract.** This paper is primarily intended as an introduction for mathematicians to some of the rich algebraic combinatorics arising in for instance conformal field theory (CFT). It tries to refine, modernise, and bridge the gap between papers [4] and [39]. Our paper is essentially self-contained, apart from some of the background motivation (Section I) and examples (Section III) which are included to give the reader a sense of the context. Detailed proofs will appear elsewhere. The theory is still a work-in-progress, and emphasis is given here to several open questions and problems.

## I. Introduction

In Segal's axioms of CFT [83], any Riemann surface with boundary is assigned a certain linear homomorphism. Roughly speaking, Borcherds [13] and Frenkel-Lepowsky-Meurman [37] axiomatised this data corresponding to a sphere with 3 disks removed, and the result is called a vertex operator algebra. Here we do the same with the data corresponding to a torus (and to a lesser extent a cylinder). The result is considerably simpler, as we shall see.

*Moonshine* in its more general sense involves the assignment of modular (automorphic) functions or forms to certain algebraic structures, e.g. theta functions to lattices, or vector-valued Jacobi forms to affine algebras, or Hauptmoduls to the Monster. This paper explores an important facet of Moonshine theory: the associated modular group representation. From this perspective, *Monstrous* Moonshine [14] is maximally uninteresting: the corresponding representation is completely trivial!

Let's focus now on the former context. Do not be put-off if this introductory section contains many unfamiliar terms. This section is motivational, supplying some of the background physical context, and many of the terms here will be mathematically addressed in later sections. It is intended to be skimmed.

A rational conformal field theory (RCFT) has two vertex operator algebras (VOAs)  $\mathcal{V}, \mathcal{V}'$ . For simplicity we will take them to be isomorphic (otherwise the RCFT is called 'heterotic'). The VOA  $\mathcal{V}$  will have finitely many irreducible modules  $A$ . Consider their (normalised) characters

$$\text{ch}_A(\tau) = q^{-c/24} \text{Tr}_A q^{L_0} \quad (1.1)$$

where  $c$  is the rank of the VOA and  $q = e^{2\pi i\tau}$ , for  $\tau$  in the upper half-plane  $\mathbb{H}$ . A VOA  $\mathcal{V}$  is (among other things) a vector space with a grading given by the eigenspaces of the

operator  $L_0$ ; (1.1) defines the character to be obtained from the induced  $L_0$ -grading on the  $\mathcal{V}$ -modules  $A$ . These characters yield a representation of the modular group  $\mathrm{SL}_2(\mathbb{Z})$  of the torus, given by its familiar action on  $\mathbb{H}$  via fractional linear transformations. In particular, we can define matrices  $S$  and  $T$  by

$$\mathrm{ch}_A(-1/\tau) = \sum_B S_{AB} \mathrm{ch}_B(\tau), \quad \mathrm{ch}_A(\tau + 1) = \sum_B T_{AB} \mathrm{ch}_B(\tau); \quad (1.2a)$$

this representation sends

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T. \quad (1.2b)$$

We call this representation the *modular data* of the RCFT. It has some interesting properties, as we shall see. For example, in Monstrous Moonshine the relevant VOA is the Moonshine module  $V^\natural$ . There is only one irreducible module of  $V^\natural$ , namely itself, and its character  $j(\tau) - 744$  is invariant under  $\mathrm{SL}_2(\mathbb{Z})$ .

Incidentally, there is in RCFT and related areas a (projective) representation of each mapping class group — see e.g. [2,43,73,85] and references therein. These groups play the role of modular group, for any Riemann surface. Their representations coming from e.g. RCFT are still poorly understood, and certainly deserve more attention, but in this paper we will consider only  $\mathrm{SL}_2(\mathbb{Z})$  (i.e. the unpunctured torus).

Strictly speaking we need linear independence of our characters, which means considering the ‘1-point functions’

$$\mathrm{ch}_A(\tau, u) = q^{-c/24} \mathrm{Tr}_A(q^{L_0} o(u))$$

— this is why  $\mathrm{SL}_2(\mathbb{Z})$  and not  $\mathrm{PSL}_2(\mathbb{Z})$  arises here — but for simplicity we will ignore this technicality in the following.

In physical parlance, the two VOAs are the (right- and left-moving) algebras of (chiral) observables. The observables operate on the space  $\mathcal{H}$  of physical states of the theory; i.e.  $\mathcal{H}$  carries a representation of  $\mathcal{V} \otimes \mathcal{V}$ . The irreducible modules  $A \otimes A'$  of  $\mathcal{V} \otimes \mathcal{V}$  in  $\mathcal{H}$  are labelled by the *primary fields* — special states  $|\phi, \phi'\rangle$  in  $\mathcal{H}$  which play the role of highest weight vectors. More precisely, the primary field will be a vertex operator  $Y(\phi, z)$  and the ground state  $|\phi\rangle$  will be the state created by the primary field at time  $t = -\infty$ :  $|\phi\rangle = \lim_{z \rightarrow 0} Y(\phi, z)|0\rangle$ . The VOA  $\mathcal{V}$  acting on the (chiral) primary field  $|\phi\rangle$  generates the module  $A = A_\phi$  (and similarly for  $\phi'$ ). The characters  $\mathrm{ch}_A$  form a basis for the vector space of 0-point 1-loop conformal blocks (see (3.7) with  $g = 1, t = 0$ ).

Modular data is a fundamental ingredient of the RCFT. It appears for instance in Verlinde’s formula (2.1), which gives (by definition) the structure constants for the fusion ring. It also constrains the torus partition function  $\mathcal{Z}$ :

$$\mathcal{Z}(\tau) = q^{-c/24} \bar{q}^{-c/24} \mathrm{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{L'_0} \quad (1.3a)$$

where  $\bar{q}$  is the complex conjugate of  $q$ . Now as mentioned above,  $\mathcal{H}$  has the decomposition

$$\mathcal{H} = \bigoplus_{A,B} M_{AB} A \otimes B \quad (1.3b)$$

into  $\mathcal{V}$ -modules, where the  $M_{AB}$  are multiplicities, and so

$$\mathcal{Z}(\tau) = \sum_{A,B} M_{AB} \operatorname{ch}_A(\tau) \overline{\operatorname{ch}_B(\tau)} \quad (1.3c)$$

Physically,  $\mathcal{Z}$  is the 1-loop vacuum-to-vacuum amplitude of the closed string (or rather, the amplitude would be  $\int \mathcal{Z}(\tau) d\tau$ ). ‘Amplitudes’ are the fundamental numerical quantities in quantum theories, from which the experimentally determinable probabilities are obtained. In Segal’s formalism, the torus  $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  is assigned the homomorphism  $\mathbb{C} \rightarrow \mathbb{C}$  corresponding to multiplication by  $\mathcal{Z}(\tau)$ . We will see in Section V that  $\mathcal{Z}$  must be invariant under the action (1.2a) of the modular group  $\operatorname{SL}_2(\mathbb{Z})$ , and so we call it (or equivalently its matrix  $M$  of multiplicities) a *modular invariant*.

Another elementary but fundamental quantity is the 1-loop vacuum-to-vacuum amplitude  $\mathcal{Z}_{\alpha\beta}$  of the open string, to whose ends are attached ‘boundary states’  $|\alpha\rangle, |\beta\rangle$  — this cylindrical partition function looks like

$$\mathcal{Z}_{\alpha\beta}(t) = \sum_A \mathcal{N}_{A\alpha}^\beta \operatorname{ch}_A(it) \quad (1.4)$$

where these multiplicities  $\mathcal{N}_{A\alpha}^\beta$  have something to do with Verlinde’s formula (2.1). These functions  $\mathcal{Z}_{\alpha\beta}$  (or equivalently their matrices  $(\mathcal{N}_A)_{\alpha\beta} = \mathcal{N}_{A\alpha}^\beta$  of coefficients) are called *fusion graphs* or *NIM-reps*, for reasons that will be explained in Section V.

We define modular invariants and NIM-reps axiomatically in Section V. Classifying them is essentially the same as classifying (boundary) RCFTs, and is an interesting and accessible challenge. All of this will be explained more thoroughly and rigourously in the course of this paper.

In this paper we survey the basic theory and examples of modular data and fusion rings. Then we sketch the basic theory of modular invariants and NIM-reps. Finally, we specialise to the modular data associated to affine Kac-Moody algebras, and discuss what is known about their modular invariant (and NIMrep) classifications. A familiarity with RCFT is not needed to read this paper (apart from this introduction!).

The theory of fusion rings (and modular data) in its purest form is the study of the algebraic consequences of requiring structure constants to obey the constraints of positivity and integrality, as well as imposing some sort of self-duality condition identifying the ring with its dual. But one of the thoughts running through this note is that we don’t know yet its correct definition. In the next section is given the most standard definition, but surely it can be improved. How to determine the correct definition is clear: we probe it from the ‘inside’ — i.e. with strange examples which we probably want to call modular data — and also from the ‘outside’ — i.e. with examples probably too dangerous to include in the fold. Some of these critical examples will be described below.

NOTATIONAL REMARKS: Throughout the paper we let  $\mathbb{Z}_{\geq}$  denote the nonnegative integers, and  $\bar{x}$  denote the complex conjugate of  $x$ . The transpose of a matrix  $A$  will be written  $A^t$ .

## II. Modular data and fusion rings

The most basic structure considered in this paper is that of modular data; the particular variant studied here — and the most common one in the literature — is given in Definition 1. But there are alternatives, and a natural general one is given by **MD1'**, **MD2'**, **MD3**, and **MD4**. In the more limited context of e.g. RCFT, the appropriate axioms are **MD1**, **MD2'**, and **MD3–MD6**.

**Definition 1.** Let  $\Phi$  be a finite set of labels, one of which — we will denote it 0 and call it the ‘identity’ — is distinguished. By *modular data* we mean matrices  $S = (S_{ab})_{a,b \in \Phi}$ ,  $T = (T_{ab})_{a,b \in \Phi}$  of complex numbers such that:

- MD1.**  $S$  is unitary and symmetric, and  $T$  is diagonal and of finite order: i.e.  $T^N = I$  for some  $N$ ;
- MD2.**  $S_{0a} > 0$  for all  $a \in \Phi$ ;
- MD3.**  $S^2 = (ST)^3$ ;
- MD4.** The numbers defined by

$$N_{ab}^c = \sum_{d \in \Phi} \frac{S_{ad} S_{bd} \overline{S_{cd}}}{S_{0d}} \quad (2.1)$$

are in  $\mathbb{Z}_{\geq}$ .

The matrix  $S$  is more important than  $T$ . The name ‘modular data’ is chosen because  $S$  and  $T$  give a representation of the (double cover of the) modular group  $\mathrm{SL}_2(\mathbb{Z})$  — as **MD3** strongly hints and as we will see in Section IV. Trying to remain consistent with the terminology of RCFT, we will call (2.1) ‘Verlinde’s formula’, the  $N_{ab}^c$  ‘fusion coefficients’, and the  $a \in \Phi$  ‘primaries’. The distinguished primary ‘0’ is called the ‘identity’ because of its role in the associated fusion ring, defined below. A possible fifth axiom will be proposed shortly, and later we will propose refinements to **MD1** and **MD2**, as well as a possible 6th axiom, but in this paper we will limit ourselves to the consequences of **MD1–MD4**.

Modular data arises in many places in math — some of these will be reviewed next section. In many of these interpretations, there is for each primary  $a \in \Phi$  a function (a ‘character’)  $\chi_a : \mathbb{H} \rightarrow \mathbb{C}$  which yields the matrices  $S$  and  $T$  as in (1.2a). Also, in many examples, to each triple  $a, b, c \in \Phi$  we get a vector space  $\mathcal{H}_{ab}^c$  (an ‘intertwiner space’ or ‘multiplicity module’) with  $\dim(\mathcal{H}_{ab}^c) = N_{ab}^c$ , and with natural isomorphisms between  $\mathcal{H}_{ab}^c$ ,  $\mathcal{H}_{ba}^c$ , etc. In many of these examples, we have ‘6j-symbols’, i.e. for any 6-tuple  $a, b, c, d, e, f \in \Phi$  we have a homomorphism  $\{ \begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix} \}$  from  $\mathcal{H}_{cd}^e \otimes \mathcal{H}_{ab}^c$  to  $\mathcal{H}_{af}^e \otimes \mathcal{H}_{bd}^f$  obeying several conditions (see e.g. [85,33] for a general treatment). Classically, 6j-symbols explicitly described the change between the two natural bases of the tensor product  $(L_\lambda \otimes L_\mu) \otimes L_\nu \cong L_\lambda \otimes (L_\mu \otimes L_\nu)$  of modules of a Lie group, and our 6j-symbols are their natural extension to e.g. quantum groups. Characters, intertwiner spaces, and 6j-symbols don’t play any role in this paper.

If **MD2** looks unnatural, think of it in the following way. It is easy to show (using **MD1** and **MD4** and Perron-Frobenius theory [55]) that some column of  $S$  is nowhere 0 and of constant phase (i.e.  $\mathrm{Arg}(S_{\uparrow b})$  is constant for some  $b \in \Phi$ ); **MD2** tells us that it is the 0 column, and that the phase is 0 (so these entries are positive). The ratios  $S_{a0}/S_{00}$  are sometimes called *q(uantum)-dimensions* (see (4.2b) below).

If **MD4** looks peculiar, think of it in the following way. For each  $a \in \Phi$ , define matrices  $N_a$  by  $(N_a)_{bc} = N_{ab}^c$ . These are usually called *fusion matrices*. Then **MD4** tells us these  $N_a$ 's are simultaneously diagonalised by  $S$ , with eigenvalues  $S_{ad}/S_{0d}$ .

The key to modular data is equation (2.1). It should look familiar from the character theory of finite groups: Let  $G$  be any finite group, let  $K_1, \dots, K_h$  be the conjugacy classes of  $G$ , and write  $k_i$  for the formal sum  $\sum_{g \in K_i} g$ . These  $k_i$ 's form a basis for the centre of the group algebra  $\mathbb{C}G$  of  $G$ . If we write

$$k_i k_j = \sum_{\ell} c_{ij\ell} k_{\ell}$$

then the structure constants  $c_{ij\ell}$  are nonnegative integers, and we obtain

$$c_{ij\ell} = \frac{\|K_i\| \|K_j\| \|K_{\ell}\|}{\|G\|} \sum_{\chi \in \text{Irr } G} \frac{\chi(g_i) \chi(g_j) \overline{\chi(g_{\ell})}}{\chi(e)}$$

where  $g_i \in K_i$ . This resembles (2.1), with  $S_{ab}$  replaced with  $S_{i,\chi} = \chi(g_i)$  and the identity 0 replaced with the group identity  $e$ . This formal relation between finite groups and Verlinde's formula seems to have first been noticed in [68].

More generally, modular data is closely related to association schemes and C-algebras (as first noted in [24], and independently in [4]), hypergroups [90], etc. That is to say, their axiomatic systems are similar. However, the exploration of an axiomatic system is influenced not merely by its intrinsic nature (i.e. its formal list of axioms and their logical consequences), but also by what are perceived by the Brethren to be its characteristic examples. There always is a context to math. The prototypical example of a C-algebra is the space of class functions of a finite group while that of modular data corresponds to the  $\text{SL}_2(\mathbb{Z})$  representation associated to an affine Kac-Moody algebra at level  $k \in \mathbb{Z}_{\geq}$  (Example 2 below). Nevertheless it can be expected that techniques and questions from one of these areas can be profitably carried over to the other. To give one interesting disparity, the commutative association schemes have been classified up to 23 vertices [59], while modular data is known for only 3 primaries [17] (and that proof assumes additional axioms)! In fact we still don't have a finiteness theorem: for a given cardinality  $\|\Phi\|$ , are there only finitely many possible modular data? But see [32] for a more sophisticated and promising approach to modular data classification.

The matrix  $T$  is fairly poorly constrained by **MD1–MD4**. Another axiom, obeyed by Examples 1,2,3 next section (as well as any conformal field theory [25]), can be introduced, though it won't be adopted here:

**MD5.** For all choices  $a, b, c, d \in \Phi$ ,

$$(T_{aa} T_{bb} T_{cc} T_{dd} T_{00}^{-1})^{N_{abcd}} = \prod_{e \in \Phi} T_{ee}^{N_{abcd,e}}$$

where

$$N_{abcd} := \sum_{e \in \Phi} N_{ab}^e N_{ce}^d, \quad N_{abcd,e} := N_{ab}^e N_{ce}^d + N_{bc}^e N_{ae}^d + N_{ac}^e N_{be}^d$$

From **MD5** can be proved that  $T$  has finite order (take  $a = b = c = d$ ), so admitting **MD5** permits us to remove that statement from **MD1**. But it doesn't have any other interesting consequences that this author knows — though perhaps it will be useful in proving the Congruence Subgroup Property given below, or give us some finiteness result.

Intimately related to modular data are the *fusion rings*  $R = \mathcal{F}(\Phi, N)$ .

**Definition 2.** A fusion ring is a commutative ring  $R$  with identity 1, together with a finite basis  $\Phi$  (over  $\mathbb{Q}$ ) containing 1, such that:

- F1.** The structure constants  $N_{ab}^c$  are all nonnegative;
- F2.** There is a ring endomorphism  $x \mapsto x^*$  stabilising the basis  $\Phi$ ;
- F3.**  $N_{ab}^1 = \delta_{b,a^*}$ .

That  $x \mapsto x^*$  is an involution is clear from **F3** and commutativity of  $R$ . Axiom **F3** and associativity of  $R$  imply  $N_{xy}^z = N_{xz}^{y^*}$  (a.k.a. Frobenius reciprocity or Poincaré duality); hence the numbers  $N_{xyz} := N_{xy}^{z^*}$  will be symmetric in  $x, y, z$ . Axiom **F3** is equivalent to the existence on  $R$  of a linear functional ‘Tr’ for which  $\Phi$  is orthonormal:  $\text{Tr}(xy^*) = \delta_{x,y} \forall x, y \in \Phi$ . Then  $N_{xy}^z = \text{Tr}(xyz^*)$ . The underlying coefficient ring was chosen to be  $\mathbb{Q}$  here, but that choice isn't important (except that it forces the coefficients  $N_{ab}^c$  to be rational).

As an abstract algebra,  $R$  is not very interesting: in particular, because  $R$  is commutative and associative, the fusion matrices  $(N_a)_{bc} = N_{ab}^c$  pairwise commute; because of **F2**,  $(N_a)^t = N_{a^*}$ . Thus they are normal and can be simultaneously diagonalised. Hence  $R$  is semisimple, and will be isomorphic to a direct sum of number fields (see Example 7 below). For example, the fusion ring for  $A_1^{(1)}$  level  $k$  (see (3.5c)) is isomorphic to  $\bigoplus_d \mathbb{Q}[\cos(\pi \frac{d}{k+2})]$ , where  $d$  runs over all divisors of  $2(k+2)$  in the interval  $1 \leq d < k+2$ . Likewise, the fusion ring  $R \otimes_{\mathbb{Q}} \mathbb{C}$  over  $\mathbb{C}$  is isomorphic as a  $\mathbb{C}$ -algebra to  $\mathbb{C}^{\|\Phi\|}$  with operations defined component-wise. Of course what is important for fusion rings is that they have a preferred basis  $\Phi$ , unlike more familiar algebras. Incidentally, more general fusion-like rings arise naturally in subfactors (see Example 6 below) and nonrational logarithmic CFT (see e.g. [48]) so their theory also should be developed.

We usually will be interested in the ‘fusion coefficients’  $N_{ab}^c$  being (nonnegative) integers. Note that the identity of fusion rings is denoted here by ‘1’ rather than the ‘0’ used in modular data.

Our treatment now will roughly follow that of Kawada's C-algebras as given in [6]. The fusion matrices  $N_a$  are linearly independent, by **F3**. Let  $\underline{x}_i$ , for  $1 \leq i \leq n = \|\Phi\|$ , be a basis of common eigenvectors, with eigenvalues  $\ell_i(a)$ . Normalise all vectors  $\underline{x}_i$  to have unit length (there remains an ambiguity of phase which we will fix below), and let  $\underline{x}_1$  be the Perron-Frobenius one — since  $\sum_a N_a > 0$  here, we can choose  $\underline{x}_1$  to be strictly positive. Let  $S$  be the matrix whose  $i$ th column is  $\underline{x}_i$ , and  $L$  the matrix  $L_{ai} = \ell_i(a)$ . Then  $S$  is unitary and  $L$  is invertible. Note that for each  $i$ , the map  $a \mapsto \ell_i(a)$  defines a linear representation of  $R$ . That means that each column of  $L$  will be a common eigenvector of all  $N_a$ , with eigenvalue  $\ell_i(a)$ , and hence must equal a scalar multiple of the  $i$ th column of  $S$  (see the BASIC FACT in Section IV). Note that each  $L_{1i} = 1$ ; therefore each  $S_{1i}$  will be nonzero and we may uniquely determine  $S$  (up to the ordering of the columns) by demanding that each  $S_{1i} > 0$ . Then  $L_{ai} = S_{ai}/S_{1i}$ . Therefore we get (2.1).

Note though that the rows of  $S$  are indexed by  $\Phi$ , but its columns are indexed by the eigenvectors. Like the character table of a group, although  $S$  is a square matrix it is not at this point in our exposition ‘truly square’. This simple observation will be valuable for the paragraph after Prop. 1.

The involution  $a \mapsto a^*$  in **F2** appears in the matrix  $C_l := SS^t$ :  $(C_l)_{ab} = \delta_{b,a^*}$ . The matrix  $C_r := S^tS$  is also an order 2 permutation, and

$$\overline{S_{ai}} = S_{C_l a, i} = S_{a, C_r i} \quad (2.2)$$

For a proof of those statements, see (4.4) below.

Let  $\widehat{R}$  be the set of all linear maps of  $R \otimes_{\mathbb{Q}} \mathbb{C}$  into  $\mathbb{C}$ , equivalently the set of all maps  $\Phi \rightarrow \mathbb{C}$ .  $\widehat{R}$  has the structure of an  $(n+1)$ -dimensional commutative algebra over  $\mathbb{C}$ , using the product  $(fg)(a) = f(a)g(a)$ . A basis  $\widehat{\Phi}$  of  $\widehat{R}$  consists of the functions  $a \mapsto \frac{S_{ai}}{S_{a1}}$ , for each  $1 \leq i \leq n$  — denote this function  $\widehat{i}$ . The resulting structure constants are

$$\widehat{N}_{\widehat{i}\widehat{j}}^k = \sum_{a \in \Phi} \frac{S_{ai} S_{aj} \overline{S_{ak}}}{S_{a1}} =: \widehat{N}_{ij}^k \quad (2.3)$$

In other words, replace  $S$  in (2.1) with  $S^t$ . It is easy to verify that  $\widehat{R} = \mathcal{F}(\widehat{\Phi}, \widehat{N})$  obeys all axioms of a fusion ring (over  $\mathbb{C}$  rather than  $\mathbb{Q}$ ), except possibly that the structure constants may not be nonnegative. They will necessarily be real, however. We call  $\widehat{R} = \mathcal{F}(\widehat{\Phi}, \widehat{N})$  the *dual* of  $R = \mathcal{F}(\Phi, N)$ . Note that  $\widehat{R}$  can always be naturally identified with  $R \otimes \mathbb{C}$ .

We call  $R = \mathcal{F}(\Phi, N)$  *self-dual* if  $\widehat{R} = \mathcal{F}(\widehat{\Phi}, \widehat{N})$  is isomorphic as a fusion ring to  $R \otimes \mathbb{C}$  — equivalently, if there is a bijection  $\iota : \Phi \rightarrow \widehat{\Phi}$  such that  $N_{ab}^c = \widehat{N}_{\iota a, \iota b}^{c \iota}$  (see the definition of ‘fusion-isomorphism’ in Section IV).

**PROPOSITION 1.** *Given any fusion ring  $R = \mathcal{F}(\Phi, N)$ , there is a unique (up to ordering of the columns) unitary matrix  $S$  obeying (2.1) and all  $S_{1i}$  and  $S_{a1}$  are positive. The fusion ring  $R = \mathcal{F}(\Phi, N)$  is self-dual iff the corresponding matrix  $S$  obeys*

$$S_{a, \iota' b} = S_{b, \iota a} \quad \text{for all } a, b \in \Phi \quad (2.4)$$

for some bijections  $\iota, \iota' : \Phi \rightarrow \widehat{\Phi}$ .

What this tells us is that there isn’t a natural algebraic interpretation for our condition  $S = S^t$  in **MD1**; the study of fusion rings insists that the definition of modular data be extended to the more general setting where ‘ $S = S^t$ ’ is replaced with (2.4). Fortunately, all properties of modular data extend naturally to this new setting. But what should  $T$  look like then? A priori this isn’t so clear. But requiring the existence of a representation of  $\mathrm{SL}_2(\mathbb{Z})$  really forces matters. In particular note that, when  $S$  is not symmetric, the matrices  $S$  and  $T$  themselves cannot be expected to give a natural representation of any group (modular or otherwise) since for instance the expression  $S^2$  really isn’t sensible —  $S$  is not ‘truly square’. Write  $P$  and  $Q$  for the matrices  $P_{a,i} = \delta_{i,\iota a}$  and  $Q_{a,i} = \delta_{i,\iota' a}$ , and let  $n$  be the order of the permutation  $\iota^{-1} \circ \iota'$ . Then for any  $k$ ,  $\tilde{S} = SQ^t(PQ^t)^k$  is ‘truly square’ and

its square  $\tilde{S}^2 = C_l(PQ^t)^k$  is a permutation matrix, where  $C_l$  is as in (2.2). We also want  $\tilde{S}^4 = I$ , which requires  $n = 2k + 1$  or  $n = 4k + 2$ . In either of those cases,  $\tilde{T} = TP^t(QP^t)^k$  defines with  $\tilde{S}$  a representation of  $\mathrm{SL}_2(\mathbb{Z})$  provided  $TST^tT = S(Q^tP)^{2k+1}$ . (When 4 divides  $n$ , the best we will get in general will be a representation of some extension of  $\mathrm{SL}_2(\mathbb{Z})$ .) But  $S$  is only determined by the fusion ring up to permutation of the columns, so we may as well replace it with  $\tilde{S}$ . Do likewise with  $T$ . So it seems that we can and should replace **MD1** with:

**MD1'.**  $S$  is unitary,  $S^t = SP$  where  $P$  is a permutation matrix of order a power of 2, and  $T$  is diagonal and of finite order;

and leave **MD2–MD4** intact. That simple change seems to provide the natural generalisation of modular data to any self-dual fusion ring. Let  $n$  be the order of  $P$ ; then  $n = 1$  recovers modular data,  $n \leq 2$  yields a representation of  $\mathrm{SL}_2(\mathbb{Z})$ , and  $n > 2$  yields a representation of a central extension of  $\mathrm{SL}_2(\mathbb{Z})$ .

If we don't require an  $\mathrm{SL}_2(\mathbb{Z})$  representation, then of course we get much more freedom. It is very unclear though what  $T$  should look like when the fusion ring is not self-dual, which probably indicates that the definition of fusion ring should include some self-duality constraint. This is the attitude we adopt.

Incidentally, the natural appearance of a self-duality constraint here perhaps should not be surprising in hindsight. Drinfeld's ‘quantum double’ construction has analogues in several contexts, and is a way of generating algebraic structures which possess modular data (see examples next section). It always involves combining a given (inadequate) algebraic structure with its dual in some way. A general categorical interpretation of quantum double is the *centre construction*, described for instance in [67]; it assigns to a tensor category a braided tensor category. It would be interesting to interpret this construction at the more base level of fusion ring — e.g. as a general way for obtaining self-dual fusion rings from non-self-dual ones.

In Example 4 of Section III we will propose a further generalisation of modular data. In this paper however, we will restrict to the consequences of **MD1–MD4**.

In any case, a fusion ring with integral fusion coefficients  $N_{ab}^c$ , self-dual in the strong sense that  $\iota = \iota'$ , is completely equivalent to a unitary and symmetric matrix  $S$  obeying **MD2**. This special case of Proposition 1 was known to Bannai and Zuber. More generally,  $\iota^{-1} \circ \iota'$  will define a *fusion-automorphism* of a self-dual fusion-ring  $R = \mathcal{F}(\Phi, N)$ . Note that an unfortunate choice of matrix  $S$  in [4] led to an inaccurate conclusion there regarding fusion rings and Verlinde's formula (2.1). In fact, Verlinde's formula will hold with a unitary matrix  $S$  obeying  $S_{1i} > 0$ , even if we drop nonnegativity **F1**.

Proposition 1 shows that although (2.1) looks mysterious, it is quite canonical, and that the depth of Verlinde's formula lies in the interpretation given to  $S$  and  $N$  (for instance (1.2a) and  $N_{ab}^c = \dim(\mathcal{H}_{ab}^c)$ ) within the given context.

The two-dimensional fusion rings  $\mathcal{F}(\{1, 2\}, N)$  are classified by their value of  $r = N_{22}^2$  — there is a unique fusion ring for every  $r \in \mathbb{Q}$ ,  $r \geq 0$ . All are self-dual. A diagonal unitary matrix  $T$  satisfying  $(ST)^3 = S^2$  exists, iff  $0 \leq r \leq \frac{2}{\sqrt{3}}$ . However,  $T$  will in addition be of finite order, i.e.  $S$  and  $T$  will constitute modular data, iff  $r = 0$  (realised e.g. by the affine algebras  $A_1^{(1)}$  and  $E_7^{(1)}$  level 1) or  $r = 1$  (realised e.g. by  $G_2^{(1)}$  and  $F_4^{(1)}$  level 1). Both  $r = 0, 1$  have six possibilities for the matrix  $T$  ( $T$  can always be multiplied by

a third root of unity). All 12 sets of modular data with two primaries can be realised by affine algebras (see Example 2 below). This seeming omnipresence of the affine algebras is an accident of small numbers of primaries; even when  $\|\Phi\| = 3$  we find non-affine algebra modular data. The (rational) fusion rings given here can be regarded as a deformation interpolating between e.g. the  $A_1^{(1)}$  and  $G_2^{(1)}$  level 1 fusion rings; similar deformations exist in higher dimensions. For example in 3-dimensions, the  $A_2^{(1)}$  level 1 fusion ring lies in a family of (rational) self-dual fusion rings parametrised by the Pythagorean triples.

*Classifying modular data and fusion rings for small sets of primaries, or at least obtaining new explicit families beyond Examples 1-3 given next section, is perhaps the most vital challenge in the theory.*

### III. Examples of modular data and fusion rings

We can find (2.1), if not modular data in its full splendor, in a wide variety of contexts. In this section we sketch several of these. Historically for the subject, Example 2 has been the most important. As with the introductory section, don't be concerned if most of these examples aren't familiar — just move on to Section IV.

EXAMPLE 1: *Lattices.* See [19] for the essentials of lattice theory.

Let  $\Lambda$  be an even lattice — i.e.  $\Lambda$  is the  $\mathbb{Z}$ -span of a basis of  $\mathbb{R}^n$ , with the property that  $x \cdot y \in \mathbb{Z}$  and  $x \cdot x \in 2\mathbb{Z}$  for all  $x, y \in \Lambda$ . Its dual  $\Lambda^*$  consists of all vectors  $w$  in  $\mathbb{R}^n$  whose dot product  $w \cdot x$  with any  $x \in \Lambda$  is an integer. So we have  $\Lambda \subset \Lambda^*$ . Let  $\Phi = \Lambda^*/\Lambda$  be the cosets. The cardinality of  $\Phi$  is finite, given by the determinant  $|\Lambda|$  of  $\Lambda$  (which equals the volume-squared of any fundamental region). The dot products  $a \cdot b$  and norms  $a \cdot a$  for the classes  $[a], [b] \in \Phi$  are well-defined (mod 1) and (mod 2), respectively. Define matrices by

$$S_{[a],[b]} = \frac{1}{\sqrt{|\Lambda|}} e^{2\pi i a \cdot b} \quad (3.1a)$$

$$T_{[a],[a]} = e^{\pi i a \cdot a - n\pi i/12} \quad (3.1b)$$

The simplest special case is  $\Lambda = \sqrt{N}\mathbb{Z}$  for any even number  $N$ , where  $\Lambda^* = \frac{1}{\sqrt{N}}\mathbb{Z}$  and  $|\Lambda| = N$ . Then  $\Phi$  can be identified with  $\{0, 1, \dots, N-1\}$ , and  $a \cdot b$  is given by  $ab/N$ , so (3.1a) becomes the finite Fourier transform.

For any such lattice  $\Lambda$ , this defines modular data. Note that the  $\mathrm{SL}_2(\mathbb{Z})$ -representation is essentially a Weil representation of  $\mathrm{SL}_2(\mathbb{Z}/|\Lambda|\mathbb{Z})$ , and that it is realised in the sense of (1.2) by characters  $\mathrm{ch}_{[a]}$  given by theta functions divided by  $\eta(\tau)^n$ . The identity '0' here is  $[0] = \Lambda$ . The fusion coefficients  $N_{[a],[b]}^{[c]}$  equal the Kronecker delta  $\delta_{[c],[a+b]}$ , so the product in the fusion ring is given by addition in  $\Lambda^*/\Lambda$ . From our point of view, this lattice example is too trivial to be interesting.

When  $\Lambda$  is merely integral (i.e. some norms  $x \cdot x$  are odd), we don't have modular data:  $T^2$  (but not  $T$ ) is defined by (3.1b), and we get a representation of  $\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle$ , an index-3 subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ . However, nothing essential is lost, so the definition of modular data should be broadened to include at minimum all these integral lattice examples.

■

EXAMPLE 2: *Kac-Moody algebras.* See [63,66] for the basics of Kac-Moody algebras.

The source of some of the most interesting modular data are the affine nontwisted Kac-Moody algebras  $X_r^{(1)}$ . The simplest way to construct affine algebras is to let  $X_r$  be any finite-dimensional simple (more generally, reductive) Lie algebra. Its loop algebra is the set of all formal series  $\sum_{\ell \in \mathbb{Z}} t^\ell a_\ell$ , where  $t$  is an indeterminant,  $a_\ell \in X_r$  and all but finitely many  $a_\ell$  are 0. This is a Lie algebra, using the obvious bracket, and is infinite-dimensional. The affine algebra  $X_r^{(1)}$  is simply a certain central extension of the loop algebra. (As usual, the central extension is taken in order to get a rich supply of representations.)

The representation theory of  $X_r^{(1)}$  is analogous to that of  $X_r$ . We are interested in the integral highest weight representations. These are partitioned into finite families parametrised by the level  $k \in \mathbb{Z}_{\geq}$ . Write  $P_+^k(X_r^{(1)})$  for the finitely many level  $k$  highest weights  $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r$ ,  $\lambda_i \geq 0$ . For example,  $P_+^k(A_r^{(1)})$  consists of the  $\binom{k+r}{r}$  such  $\lambda$ , which obey  $\lambda_0 + \lambda_1 + \cdots + \lambda_r = k$ .

The  $X_r^{(1)}$ -character  $\chi_\lambda(\tau)$  associated to highest weight  $\lambda$  is given by a graded trace, as in (1.1). Thanks to the structure and action of the affine Weyl group on the Cartan subalgebra of  $X_r^{(1)}$ , the character  $\chi_\lambda$  is essentially a lattice theta function, and so transforms nicely under the modular group  $SL_2(\mathbb{Z})$ . In fact, for fixed algebra  $X_r^{(1)}$  and level  $k \in \mathbb{Z}_{\geq}$ , these  $\chi_\lambda$  define a representation of  $SL_2(\mathbb{Z})$ , exactly as in (1.2) above, and the matrices  $S$  and  $T$  constitute modular data. The ‘identity’ is  $0 = k\Lambda_0$ , and the set of ‘primaries’ is the highest weights  $\Phi = P_+^k(X_r^{(1)})$ . The matrix  $T$  is related to the values of the second Casimir of  $X_r$ , and  $S$  to elements of finite order in the Lie group corresponding to  $X_r$ :

$$T_{\lambda\mu} = \alpha \exp\left[\frac{\pi i(\lambda + \rho|\lambda + \rho)}{\kappa}\right] \delta_{\lambda,\mu} \quad (3.2a)$$

$$S_{\mu\nu} = \alpha' \sum_{w \in \overline{W}} \det(w) \exp\left[-2\pi i \frac{(w(\mu + \rho)|\nu + \rho)}{\kappa}\right] \quad (3.2b)$$

$$\frac{S_{\lambda\mu}}{S_{0\mu}} = \text{ch}_{\overline{\lambda}}\left(\exp\left[-2\pi i \frac{(\overline{\lambda}|\overline{\mu} + \rho)}{\kappa}\right]\right) \quad (3.2c)$$

The numbers  $\alpha, \alpha' \in \mathbb{C}$  are normalisation constants whose precise values are unimportant here, and are given in Thm. 13.8 of [63]. The inner product in (3.2) is the usual Killing form,  $\rho$  is the Weyl vector  $\sum_i \Lambda_i$ , and  $\kappa = k + h^\vee$ , where  $h^\vee$  is the dual Coxeter number ( $= r + 1$  for  $A_r^{(1)}$ ). The (finite) Weyl group  $\overline{W}$  of  $X_r$  acts on  $P_+^k$  by fixing  $\Lambda_0$ . Here,  $\overline{\lambda}$  denotes the projection  $\lambda_1\Lambda_1 + \cdots + \lambda_r\Lambda_r$ , and ‘ $\text{ch}_{\overline{\lambda}}$ ’ is a finite-dimensional Lie group character.

The combinatorics of Lie group characters at elements of finite order, i.e. the ratios (3.2c), is quite rich. For example, in [62] they are used to prove quadratic reciprocity, while [72] uses them for instance in a fast algorithm for computing tensor product decompositions in Lie groups.

The fusion coefficients  $N_{\lambda\mu}^\nu$ , defined by (2.1), are essentially the tensor product multiplicities  $T_{\lambda\mu}^\nu := \text{mult}_{\overline{\lambda} \otimes \overline{\mu}}(\overline{\nu})$  for  $X_r$  (e.g. the Littlewood-Richardson coefficients for  $A_r$ ),

except ‘folded’ in a way depending on  $k$ . This is seen explicitly by the Kac-Walton formula [63 p. 288, 88,44]:

$$N_{\lambda\mu}^\nu = \sum_{w \in W} \det(w) T_{\lambda\mu}^{w.\nu}, \quad (3.3)$$

where  $w.\gamma := w(\gamma + \rho) - \rho$  and  $W$  is the affine Weyl group of  $X_r^{(1)}$  (the dependence on  $k$  arises through this action of  $W$ ). The proof of (3.3) follows quickly from (3.2c).

The fusion ring  $R$  here is isomorphic to  $\text{Ch}(X_\ell)/\mathcal{I}_k$ , where  $\text{Ch}(X_\ell)$  is the character ring of  $X_\ell$  (which is isomorphic as an algebra to the polynomial algebra in  $\ell$  variables), and where  $\mathcal{I}_k$  is its ideal generated by the characters of the ‘level  $k+1$ ’ weights (for  $X_\ell = A_\ell$ , these consist of all  $\bar{\lambda} = (\lambda_1, \dots, \lambda_\ell)$  obeying  $\lambda_1 + \dots + \lambda_\ell = k+1$ ).

Equation (3.3) has the flaw that, although it is manifest that the  $N_{\lambda\mu}^\nu$  will be integral, it is not clear why they are positive. A big open challenge here is the discovery of a combinatorial rule, e.g. in the spirit of the well-known Littlewood-Richardson rule, for the affine fusions. Three preliminary steps in this direction are [82,84,35].

Identical numbers  $N_{\lambda\mu}^\nu$  appear in several other contexts. For instance, Finkelberg [36] proved that the affine fusion ring is isomorphic to the K-ring of Kazhdan-Lusztig’s category  $\tilde{\mathcal{O}}_{-k}$  of level  $-k$  integrable highest weight  $X_r^{(1)}$ -modules, and to Gelfand-Kazhdan’s category  $\tilde{\mathcal{O}}_q$  coming from finite-dimensional modules of the quantum group  $U_q(X_r)$  specialised to the root of unity  $q = \exp[i\pi/m\kappa]$  for appropriate choice of  $m \in \{1, 2, 3\}$ . Because of these isomorphisms, we know that the  $N_{\lambda\mu}^\nu$  do indeed lie in  $\mathbb{Z}_{\geq}$ , for any affine algebra. We also know [38] that they increase with  $k$ , with limit  $T_{\lambda\mu}^\nu$ .

Also, they arise as dimensions of spaces of generalised theta functions [34], as tensor product coefficients in quantum groups [44] and Hecke algebras [58] at roots of 1 and Chevalley groups for  $\mathbb{F}_p$  [56], and in quantum cohomology [91].

For an explicit example, consider the simplest affine algebra  $(A_1^{(1)})$  at level  $k$ . We may take  $P_+^k = \{0, 1, \dots, k\}$  (the value of  $\lambda_1$ ), and then the  $S$  and  $T$  matrices and fusion coefficients are given by

$$S_{ab} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(a+1)(b+1)}{k+2}\right) \quad (3.5a)$$

$$T_{aa} = \exp\left[\frac{\pi i(a+1)^2}{2(k+2)} - \frac{\pi i}{4}\right] \quad (3.5b)$$

$$N_{ab}^c = \begin{cases} 1 & \text{if } c \equiv a+b \pmod{2} \text{ and } |a-b| \leq c \leq \min\{a+b, 2k-a-b\} \\ 0 & \text{otherwise} \end{cases} \quad (3.5c)$$

The only other affine algebras for which the fusions have been explicitly calculated are  $A_2^{(1)}$  [9] and  $A_3^{(1)}$  [10], and their formulas are also surprisingly compact.

Incidentally, an analogous modular transformation matrix  $S$  to (3.2b) exists for the so-called *admissible representations* of  $X_r^{(1)}$  at fractional level [65]. The matrix is symmetric, but has no column of constant phase and thus naively putting it into Verlinde’s formula (2.1) will necessarily produce some negative numbers (apparently they’ll always be integers though). A legitimate fusion ring has been obtained for  $A_1^{(1)}$  at fractional level  $k = \frac{p}{q} - 2$  in other ways [3]; it factorises into the product of the  $A_{1,p-2}$  fusion ring with a fusion ring

at ‘level’  $q - 1$  associated to the rank 1 supersymmetric algebra  $\text{osp}(1|2)$ . Some doubt however on the relevance of these efforts has been cast by [46]. A similar theory should exist at least for the other  $A_r^{(1)}$ ; initial steps for  $A_2^{(1)}$  have been made in [45]. Exactly how these correspond to modular data, or rather how modular data should be generalised to accommodate them, is not completely understood at this time. ■

**EXAMPLE 3:** *Finite groups.* The relevant aspects of finite group theory are given in e.g. [61].

Let  $G$  be any finite group. Let  $\Phi$  be the set of all pairs  $(a, \chi)$ , where the  $a$  are representatives of the conjugacy classes of  $G$  and  $\chi$  is the character of an irreducible representation of the centraliser  $C_G(a)$ . (Recall that the conjugacy class of an element  $a \in G$  consists of all elements of the form  $g^{-1}ag$ , and that the centraliser  $C_G(a)$  is the set of all  $g \in G$  commuting with  $a$ .) Put [26,71]

$$S_{(a,\chi),(a',\chi')} = \frac{1}{\|C_G(a)\| \|C_G(a')\|} \sum_{g \in G(a,a')} \overline{\chi'(g^{-1}ag)} \overline{\chi(ga'g^{-1})} \quad (3.6a)$$

$$T_{(a,\chi),(a',\chi')} = \delta_{a,a'} \delta_{\chi,\chi'} \frac{\chi(a)}{\chi(e)} \quad (3.6b)$$

where  $G(a, a') = \{g \in G \mid aga'g^{-1} = ga'g^{-1}a\}$ , and  $e \in G$  is the identity. For the ‘identity’ 0 take  $(e, 1)$ . Then (3.6) is modular data. See [22] for several explicit examples.

There are group-theoretic descriptions of the fusion coefficient  $N_{(a,\chi),(b,\chi')}^{(c,\chi'')}$ . That these fusion coefficients are nonnegative integers, follows for instance from Lusztig’s interpretation of the corresponding fusion ring as the Grothendieck ring of equivariant vector bundles over  $G$ :  $\Phi$  can be identified with the irreducible vector bundles.

This class of modular data played an important role in Lusztig’s determination of irreducible characters of Chevalley groups. But there is a remarkable variety of contexts in which (3.6) appears (these are reviewed in [22]). For instance, modular data often has a Hopf algebra interpretation: just as the affine fusions are recovered from the quantum group  $U_q(X_r)$ , so are these finite group fusions recovered from the quantum-double of  $G$ .

This modular data is quite interesting for nonabelian  $G$ , and deserves more study. It behaves very differently than the affine data [22]. Conformal field theory explains how very general constructions (Goddard-Kent-Olive and orbifold) build up modular data from combinations of affine and finite group data — see e.g. [25].

For a given finite group  $G$ , there doesn’t appear to be a natural unique choice of characters  $\text{ch}_{(a,\chi)}$  realising this modular data in the sense of (1.2).

This modular data can be twisted [27] by a 3-cocycle  $\alpha \in H^3(G, \mathbb{C}^\times)$ , which plays the same role here that level did in Example 2. A further major generalisation of this finite group data will be discussed in Example 6 below, and of this cohomological twist  $\alpha$  in the paragraph after Example 6. ■

**EXAMPLE 4:** *RCFT, TFT.* See e.g. [25] and [85], and references therein, for good surveys of 2-dimensional conformal and 3-dimensional topological field theories, respectively. In [39] can be found a survey of fusion rings in rational conformal field theory (RCFT).

As discussed earlier, a major source of modular data comes from RCFT (and string theory) and, more or less the same thing, 3-dimensional topological field theory (TFT).

In RCFT, the elements  $a \in \Phi$  are called ‘primary fields’, and the privileged one ‘0’ is called the ‘vacuum state’. The entries of  $T$  are interpreted in RCFT to be  $T_{aa} = \exp[2\pi i(h_a - \frac{c}{24})]$ , where  $c$  is the rank of the VOA or the ‘central charge’ of the RCFT, and  $h_a$  is the ‘conformal weight’ or  $L_0$ -eigenvalue of the primary field  $a$ . Equation (2.1) is a special case of the so-called *Verlinde’s formula* [87]:

$$V_{a^1 \dots a^t}^{(g)} = \sum_{b \in \Phi} (S_{0b})^{2(1-g)} \frac{S_{a^1 b}}{S_{0b}} \dots \frac{S_{a^t b}}{S_{0b}} \quad (3.7)$$

It arose first in RCFT as an extremely useful expression for the dimensions of the space of conformal blocks on a genus  $g$  surface with  $t$  punctures, labelled with primaries  $a^i \in \Phi$  — the fusions  $N_{ab}^c$  correspond to a sphere with 3 punctures. All the  $V^{(g)}$ ’s are nonnegative integers iff all the  $N_{ab}^c$ ’s are. In RCFT, our unused axiom **MD5** is derived by applying Dehn twists to a sphere with 4 punctures to obtain an  $N_{abcd} \times N_{abcd}$  matrix equation on the corresponding space of conformal blocks; **MD5** is the determinant of that equation [86].

Example 1 corresponds to the string theory of  $n$  free bosons compactified on the torus  $\mathbb{R}^n/\Lambda$ . Example 2 corresponds to Wess-Zumino-Witten RCFT [57] where a closed string lives on a Lie group manifold. Example 3 corresponds to the untwisted sector in an orbifold of a holomorphic RCFT (a holomorphic theory has trivial modular data — e.g. a lattice theory when the lattice  $\Lambda = \Lambda^*$  is self-dual) by  $G$  [26]. The RCFT interpretation of fractional level affine algebra modular data isn’t understood yet, despite considerable effort (see e.g. [46]).

Actually, in a *nonunitary* RCFT, the matrices  $S$  and  $T$  defined by (1.2a) will obey **MD1**, **MD3**, and **MD4**, but not **MD2**. For example, the ‘ $c = c(7, 2) = -\frac{68}{7}$  nonunitary minimal model’ has  $S$  and  $T$ , defined by (1.2a), given by

$$\begin{aligned} T &= \text{diag}\{\exp[17\pi i/21], \exp[5\pi i/21], \exp[-\pi i/21]\} \\ S &= \frac{2}{\sqrt{7}} \begin{pmatrix} \sin(2\pi/7) & -\sin(3\pi/7) & \sin(\pi/7) \\ -\sin(3\pi/7) & -\sin(\pi/7) & \sin(2\pi/7) \\ \sin(\pi/7) & \sin(2\pi/7) & \sin(3\pi/7) \end{pmatrix} \end{aligned} \quad (3.8a)$$

This is not modular data, since the first column is not strictly positive. However the 3rd column is. The nonunitary RCFTs tell us to replace **MD2** with

**MD2’.** For all  $a \in \Phi$ ,  $S_{0,a}$  is a nonzero real number. Moreover there is some  $0' \in \Phi$  such that  $S_{0',a} > 0$  for all  $a \in \Phi$ .

Incidentally, an  $S$  matrix which the algorithm of Section II would associate to that  $c = -\frac{68}{7}$  minimal model is

$$S = \frac{2}{\sqrt{7}} \begin{pmatrix} \sin(\pi/7) & \sin(2\pi/7) & \sin(3\pi/7) \\ \sin(2\pi/7) & -\sin(3\pi/7) & \sin(\pi/7) \\ \sin(3\pi/7) & \sin(\pi/7) & -\sin(2\pi/7) \end{pmatrix} \quad (3.8b)$$

We can tell by looking at (3.8b) that it can't directly be given the familiar interpretation (1.2a). The reason is that any such matrix  $S$  must have a strictly positive eigenvector with eigenvalue 1: namely the eigenvector with  $a$ th component  $\text{ch}_a(i)$  ( $\tau = i$  corresponds to  $q = e^{-2\pi} > 0$  and is fixed by  $\tau \mapsto -1/\tau$ ; moreover the characters of VOAs converge at any  $\tau \in \mathbb{H}$  [92]). Unlike the  $S$  in (3.8a), the  $S$  of (3.8b) has no such eigenvector. Thus we may find it convenient (especially in classification attempts) to introduce a new axiom:

**MD6.**  $S$  has a strictly positive eigenvector  $\underline{x} > 0$  with eigenvalue 1.

Note that with the choice  $T = \text{diag}\{\exp[\pi i/21], \exp[-17\pi i/21], \exp[-5\pi i/21]\}$ , (3.8b) obeys **MD1-MD4**. Remarkably, all nonunitary RCFT known to this author behave similarly. In fact, the following refinement of **MD2'** appears to be true: The primary  $0'$  in **MD2'** equals  $J\sigma 0$  for some simple-current  $J$  and some Galois automorphism  $\sigma$ . (The term ‘simple-current’ and the Galois action on  $\Phi$  will be defined in Section IV.) For example, in (3.8a) the simple-current is the identity and the Galois automorphism corresponds to  $5 \in (\mathbb{Z}/42\mathbb{Z})^\times$ . Whenever this refinement holds (which may be always), one consequence will be that the fusion ring is realised by modular data satisfying **MD1-MD4**.

Knot and link invariants in  $S^3$  (equivalently,  $\mathbb{R}^3$ ) can be obtained from an  $R$  matrix and braid group representations — e.g. we have this with any quasitriangular Hopf algebra. The much richer structure of *topological field theory* (or, in category theoretic language, a *modular category* [85]) gives us link invariants in any closed 3-manifold, and with it modular data. In particular, the  $S$  entries correspond to the invariants of the Hopf link in  $S^3$ ,  $T$  to the eigenvalues of the twist operation (Reidemeister 1, which won't act trivially here — strictly speaking, we have knotted ribbons, not strings), and the fusion coefficients to the invariants of 3 parallel circles  $S^1 \times \{p_1, p_2, p_3\}$  in the manifold  $S^1 \times S^2$ . Link invariants are obtained for arbitrary closed 3-manifolds by performing Dehn surgery, transforming the manifold into  $S^3$ ; the condition that the resulting invariants be well-defined, independent of the specific Dehn moves which get us to  $S^3$ , is essentially the statement that  $S$  and  $T$  form a representation of  $\text{SL}_2(\mathbb{Z})$ . This is all discussed very clearly in [85]. For instance, we get  $S^3$  knot invariants from the quantum group  $U_q(X_r)$  with generic parameter, but to get modular data requires specialising  $q$  to a root of unity.

For extensions of this picture to representations of higher genus mapping class groups, see e.g. [43] and references therein, but there is much more work to do here. ■

**EXAMPLE 5: VOAs.** See e.g. [37,64] for the basic facts about VOAs; the review article [46] illustrates how VOAs naturally arise in CFT.

Another very general source of modular data comes from vertex operator algebras (VOAs), a rich algebraic structure first introduced by Borcherds [13]. In particular, let  $\mathcal{V}$  be any ‘rational’ VOA (see e.g. [92] — actually, VOA theory is still sufficiently undeveloped that we don't yet have a generally accepted definition of rational VOA). Then  $\mathcal{V}$  will have finitely many irreducible modules  $M$ , one of which can be identified with  $\mathcal{V}$ . Zhu [92] showed that their characters  $\text{ch}_M(\tau)$  transform nicely under  $\text{SL}_2(\mathbb{Z})$  (as in (1.2a)). Defining  $S$  and  $T$  in that way, and calling  $\Phi$  the set of irreducible  $M$  and the ‘identity’  $0 = \mathcal{V}$ , we get some of the properties of modular data.

A natural conjecture is that a large class (all?) of rational VOAs possess (some generalisation of) modular data. We know what the fusion coefficients mean (dimension of the space of intertwiners between the appropriate VOA modules), and what  $S$  and  $T$  should

mean. We know that  $T$  is diagonal and of finite order, and that  $S^2 = (ST)^3$  is an order-2 permutation matrix. A Holy Grail of VOA theory is to prove (a generalisation of) Verlinde's formula for a large class of rational VOAs. A problem is that we still don't know when (2.1) here is even defined (i.e. whether all  $S_{0,M} \neq 0$ ). However, suppose  $\mathcal{V}$  has the additional (natural) property that any irreducible module  $M \neq \mathcal{V}$  has positive conformal weight  $h_M$  ( $h_M - c/24$  is the smallest power of  $q$  in the Fourier expansion of the (normalised) character  $\text{ch}_M(\tau) = q^{-c/24} \sum_{n=0}^{\infty} a_n^M q^{n+h_M}$ ). This holds for instance in all VOAs associated to unitary RCFTs. Then consider the behaviour of  $\text{ch}_M(\tau)$  for  $\tau \rightarrow 0$  along the positive imaginary axis: since each Fourier coefficient  $a_n^M$  is a nonnegative number,  $\text{ch}_M(\tau)$  will go to  $+\infty$ . But this is equivalent to considering the limit of  $\sum_N S_{MN} \text{ch}_N(\tau)$  as  $\tau \rightarrow i\infty$  along the positive imaginary axis. By hypothesis, this latter limit is dominated by  $S_{M0} a_0^M q^{-c/24}$ , at least when  $S_{M0} \neq 0$ . So what we find is that, under this hypothesis, the 0-column of  $S$  consists of nonnegative real numbers (and also that the rank  $c$  is positive).

In this context, Example 1 corresponds to the VOA associated to the lattice  $\Lambda$  [28]. Example 2 is recovered by [38], who find a VOA structure on the highest weight  $X_r^{(1)}$ -module  $L(k\Lambda_0)$ ; the other level  $k$   $X_r^{(1)}$ -modules  $M = L(\lambda)$  all have the structure of VOA modules of  $\mathcal{V} := L(k\Lambda_0)$ . Example 3 arises for example in the orbifold of a self-dual lattice VOA by a subgroup  $G$  of the automorphism group of  $\Lambda$  (see e.g. [31]). An interpretation of fractional level affine algebra data could be possible along the lines of [30], who did it for  $A_1^{(1)}$  (but see [46]). ■

**EXAMPLE 6: Subfactors.** See e.g. [33,12] for good reviews of the subfactor  $\leftrightarrow$  CFT relation.

The final general source of modular data which we will discuss comes from subfactor theory. To start with, let  $N \subset M$  be an inclusion of  $\text{II}_1$  factors with finite Jones index  $[M : N]$ . Even though  $M$  and  $N$  will often be isomorphic as factors, Jones showed that there is rich combinatorics surrounding how  $N$  is embedded in  $M$ . Write  $M_{-1} = N \subset M = M_0 \subset M_1 \subset \dots$  for the tower arising from the 'basic construction'. Let  $\Phi_M$  denote the set of equivalence classes of irreducible  $M - M$  submodules of  $\oplus_{n \geq 0} {}_M L^2(M_n)_M$ , and  $\Phi_N$  that for the irreducible  $N - N$  submodules of  $\oplus_{n \geq -1} {}_N L^2(M_n)_N$ . Write  $\mathcal{H}_{AB}^C$  for the intertwiner space  $\text{Hom}_{M-M}(C, A \otimes_M B)$ . For any  $A, B \in \Phi_M$ , the Connes' relative tensor product  $A \otimes_M B$  can be decomposed into a direct sum  $\sum_{C \in \Phi_M} N_{AB}^C C$ , where  $N_{AB}^C = \dim \mathcal{H}_{AB}^C \in \mathbb{Z}_{\geq}$  are the multiplicities. The identity is the bimodule  ${}_M L^2(M)_M$ . Assume in addition that  $\Phi_M$  is finite (i.e. that  $N \subset M$  has 'finite depth'). Then all axioms of a fusion ring will be obeyed, except possibly commutativity: unfortunately in general  $A \otimes_M B \neq B \otimes_M A$ .

We are interested in  $M$  and  $N$  being hyperfinite. An intricate subfactor invariant called a *paragroup* (see e.g. [75,33]) can be formulated in terms of 6j-symbols and fusion rings [33], and resembles exactly solvable lattice models in statistical mechanics. One way to get modular data is by passing from  $N \subset M$  to the asymptotic inclusion  $\langle M, M' \cap M_\infty \rangle \subset M_\infty$ ; its paragroup will essentially be an RCFT. Asymptotic inclusion plays the role of quantum-double here, and corresponds physically to taking the continuum limit of the lattice model, yielding the CFT from the underlying statistical mechanical model. More recently [76], Ocneanu has significantly refined this construction, generalising 6j-symbols to what are called Ocneanu cells, and extending the context to subparagroups. His new

cells have been interpreted by [77] in terms of Moore-Seiberg-Lewellen data [73,70].

A very similar but simpler theory has been developed for type III factors. Bimodules now are equivalent to ‘sectors’, i.e. equivalence classes of endomorphisms  $\lambda : N \rightarrow N$  (the corresponding subfactor is  $\lambda(N) \subset N$ ). This use of endomorphisms is the key difference (and simplification) between the type II and type III fusion theories. Given  $\lambda, \mu \in \text{End}(N)$ , we define  $\langle \lambda, \mu \rangle$  to be the dimension of the vector space of intertwiners, i.e. all  $t \in N$  such that  $t\lambda(n) = \mu(n)t \forall n \in N$ . The endomorphism  $\lambda \in \text{End}(N)$  is irreducible if  $\langle \lambda, \lambda \rangle = 1$ . Let  $\Phi = {}_N\chi_N$  be a finite set of irreducible sectors. The fusion product is given by composition  $\lambda \circ \mu$ ; addition can also be defined, and the fusion coefficient  $N_{\lambda\mu}^\nu$  will then be the dimension  $\langle \lambda\mu, \nu \rangle$ . The ‘identity’ 0 is the identity  $\text{id}_N$ . Restricting to a finite set  $\Phi$  of irreducible sectors, closed under fusion in the obvious way, the result is similar to a fusion ring, except again it is not necessarily commutative (after all, why should the compositions  $\lambda \circ \mu$  and  $\mu \circ \lambda$  be related). The missing ingredients are nondegenerate braidings  $\epsilon^\pm(\lambda, \mu) \in \text{Hom}(\lambda\mu, \mu\lambda)$ , which say roughly that  $\lambda$  and  $\mu$  nearly commute (the  $\epsilon^\pm$  must also obey some compatibility conditions, e.g. the Yang-Baxter equations). Once we have a nondegenerate braiding, Rehren [78] proved that we will automatically have modular data.

We will return to subfactors in Section V. It is probably too optimistic to hope to see in the subfactor picture to what the characters (1.1) correspond — different VOAs or RCFTs can correspond it seems to equivalent subfactors. To give a simple example, the VOA associated to any self-dual lattice will correspond to the trivial subfactor  $N = M$ , where  $M$  is the unique hyperfinite  $\text{II}_1$  factor. With this in mind, it would be interesting to find an  $S$  matrix arising here which violates axiom **MD6** given earlier, or the Congruence Subgroup Property of Section IV.

Jones and Wassermann have explicitly constructed the affine algebra subfactors (both type II and III) of Example 2, at least for  $A_r^{(1)}$ , and Wassermann and students Loke and Toledano Laredo later showed that they recover the affine algebra fusions (see e.g. [89] for a review). Also, to any subgroup-group pair  $H < G$ , we can obtain a subfactor  $R \rtimes H \subset R \rtimes G$  of crossed products, where  $R$  is the type  $\text{II}_1$  hyperfinite factor, and thus a (not necessarily commutative) fusion-like ring [69]. This subfactor  $R \rtimes H \subset R \rtimes G$  can be thought of as giving a grouplike interpretation to  $G/H$  even when  $H$  is not normal. Sometimes it will have a braiding — e.g. the diagonal embedding  $G < G \times G$  recovers the finite group data of Example 3. What is intriguing is that some other pairs  $H < G$  probably also have a braiding, generalising Example 3. There is a general suspicion, due originally perhaps to Moore and Seiberg [73] and in the spirit of Tannaka-Krein duality, that RCFTs can always be constructed in standard ways (Goddard-Kent-Olive cosets and finite group orbifolds) from lattice and affine algebra models. These crossed product subfactors could conceivably provide reams of counterexamples, suggesting that the orbifold construction can be considerably generalised. ■

A uniform construction of the affine algebra and finite group modular data is provided in [27] where a 3-dimensional TFT is associated to any topological group  $G$  ( $G$  will be a compact Lie group in the affine case;  $G$  is given discrete topology in the finite case). There we see that the level  $k$  and twist  $\alpha$  both play the same role, and are given by a cocycle in  $H^3(G, \mathbb{C}^\times)$ . Crane-Yetter [23] are developing a theory of cohomological ‘deformations’ of

modular data (more precisely, of modular categories). In [23] they discuss the infinitesimal deformations of tensor categories, where the objects are untouched but the arrows are deformed, though their ultimate interest would be in global deformations and in particular in specialising to the especially interesting ones — much as we deform the enveloping algebra  $U(\mathfrak{g})$  to get the quantum group  $U_q(\mathfrak{g})$  and then specialise to roots of unity to get e.g. modular data. Their work is still in preliminary stages and it probably needs to be generalised further (e.g. they don't seem to recover the level of affine algebras), but it looks very promising. Ultimately it can be hoped that some discrete  $H^3$  group will be identified which parametrises the different quantum doubles of a given symmetric tensor category.

Incidentally, the fact that  $H^3(G, \mathbb{C}^\times)$  is a group strongly suggests that it should be meaningful to compare the modular data for different cocycles — e.g. to fix the affine algebra and vary  $k$ . This idea still hasn't been seriously exploited (but e.g. see ‘threshold level’ in [9,10]).

There are many examples of ‘pseudo-modular data’. These are interesting for probing the question of just what should be the definition of fusion ring or modular data. Here is an intriguing example, inspired by (4.4) below.

**EXAMPLE 7** [52]: *Number fields.* A basic introduction to algebraic number theory is provided by e.g. [18].

Choose any finite normal extension  $\mathbb{L}$  of  $\mathbb{Q}$ , and find any totally positive  $\alpha \in \mathbb{L}$  with  $\text{Tr}(|\alpha|^2) = 1$  (total positivity will turn out to be necessary for **F1**). Now find any  $\mathbb{Q}$ -basis  $x_1 = 1, x_2, \dots, x_n$  of a subfield  $\mathbb{K}$  of  $\mathbb{L}$ , where  $n = \deg(\mathbb{K})$ , the  $x_i$  being orthonormal with respect to the trace  $\langle x, y \rangle_\alpha := \text{Tr}(|\alpha|^2 x \bar{y})$  (orthonormality will guarantee **F3** to be satisfied). Let  $G$  denote the set of  $n$  distinct embeddings  $\mathbb{K} \rightarrow \mathbb{C}$ . Our construction requires complex conjugation to commute with all embeddings. Under these conditions  $|\alpha|^{-2} = \sum_i |x_i|^2$ . Then we get a fusion-like ring with primaries  $\Phi = \{x_1, \dots, x_n\}$ , ‘ $*$ ’ given by complex conjugation, and structure constants  $N_{ij}^k = \text{Tr}(|\alpha|^2 x_i x_j \bar{x}_k) \in \mathbb{Q}$  given by ordinary multiplication and addition:  $x_i x_j = \sum_k N_{ij}^k x_k$ . Call the resulting fusion-like ring  $\mathbb{K}(\Phi)$ .

It is easy to see that all the properties of a fusion ring are satisfied, except possibly  $N_{ij}^k \in \mathbb{Q}_{\geq}$ . The fusion coefficients  $N_{ij}^k$  will be integers iff the  $\mathbb{Z}$ -span of the  $x_i$  form an ‘order’ of  $\mathbb{K}$ . We also find that the matrix  $S_{ig} = g(\alpha x_i)$ , for  $g \in G$  (lift each  $g$  arbitrarily to  $\mathbb{L}$ ), diagonalises these fusion matrices  $N_{x_i}$ . This matrix  $S$  is unitary, but (unless  $\mathbb{K}$  is an abelian extension of  $\mathbb{Q}$ ) the dual fusions  $\hat{N}$  in (2.3) won’t be rational.

Positivity **F1** requires one of the columns of  $S$  to be positive; permuting with  $g$ , we may require all basis elements  $x_i > 0$ . Hence  $\mathbb{K}(\Phi)$  will have a chance of being a fusion ring only when  $\mathbb{K}$  is ‘totally real’.

Incidentally this example is more general than it looks: it is easy to show

**PROPOSITION 2.** *Let  $R$  be a fusion ring which is isomorphic as a  $\mathbb{Q}$ -algebra to a field  $\mathbb{K}$ . Then  $R$  is isomorphic as a fusion ring to some  $\mathbb{K}(\Phi)$ .*

More generally, recall that an arbitrary fusion ring (over  $\mathbb{Q}$ ) is isomorphic as an algebra to a direct sum of number fields. So an approach to studying fusion rings could be to study how they are built up from number fields. It would be very interesting to classify all (not necessarily self-dual) fusion rings which are isomorphic as a  $\mathbb{Q}$ -algebra to a field. For

example, take  $\mathbb{K} = \mathbb{Q}[\sqrt{N}]$ , where  $N$  is not a perfect square, and where also any prime divisor  $p \equiv -1 \pmod{4}$  of  $N$  occurs with even multiplicity. Then we can find positive integers  $a, b$  such that  $N = a^2 + b^2$ . Take  $\Phi = \{1, \frac{b}{a} + \frac{1}{a}\sqrt{N}\}$ , then  $\mathbb{K}(\Phi)$  is a fusion ring with  $N_{22}^2 = \frac{2b}{a}$ . Note that this construction exhausts all 2-dimensional rational fusion rings, except when  $\sqrt{(N_{22}^2)^2 + 4}$  is rational, which corresponds to the  $\mathbb{Q}$ -algebra  $\mathbb{Q} \oplus \mathbb{Q}$  (e.g. the fusion ring of  $A_1^{(1)}$  level 1). For  $N = 5$  and  $a = 2$ , we recover the fusion ring of  $F_4^{(1)}$  or  $G_2^{(1)}$  level 1. ■

#### IV. Modular data: basic theory

In this section we sketch the basic theory of modular data.

It is important to reinterpret (2.1) in matrix form. For each  $a \in \Phi$ , define the *fusion matrix*  $N_a$  by

$$(N_a)_{b,c} = N_{ab}^c .$$

Then (2.1) says that the  $N_a$  are simultaneously diagonalised by  $S$ . More precisely, the  $b$ th column  $S_{\uparrow,b}$  of  $S$  is an eigenvector of each  $N_a$ , with eigenvalue  $\frac{S_{ab}}{S_{0b}}$ . Unitarity of  $S$  tells us:  $\frac{S_{ab}}{S_{0b}} = \frac{S_{ac}}{S_{0c}}$  holds for all  $a \in \Phi$ , iff  $b = c$ . In other words:

**BASIC FACT.** *All simultaneous eigenspaces are of dimension 1, and are spanned by each column  $S_{\uparrow,b}$ .*

Take the complex conjugate of (2.1): we find that  $\overline{S}$  also simultaneously diagonalises the fusion matrices  $N_a$ . Hence there is some permutation of  $\Phi$ , which we will denote by  $C$  and call *conjugation*, and some complex numbers  $\alpha_b$ , such that

$$\overline{S_{ab}} = \alpha_b S_{a,Cb} .$$

Unitarity forces each  $|\alpha_b| = 1$ . Looking at  $a = 0$  and applying **MD2**, we see that the  $\alpha_b$  must be positive. Hence

$$\overline{S_{ab}} = S_{a,Cb} = S_{Ca,b} \tag{4.1}$$

and so  $C = S^2$ . The conjugation  $C$  is trivial iff  $S$  is real. Note also that  $C$ , like complex conjugation, is an involution, and that  $C_{00} = 1$ . Some easy formulae are  $N_0 = I$ ,  $N_{ab}^0 = C_{ab}$ , and  $N_{Ca,Cb}^{Cc} = N_{ab}^c$ . Because  $C = S^2 = (ST)^3$ ,  $C$  commutes with both  $S$  and  $T$ :  $S_{Ca,Cb} = S_{a,b}$  and  $T_{Ca,Cb} = T_{a,b}$ .

For example, in Example 1,  $C[a] = [-a]$ , while for  $A_1^{(1)}$  the matrix  $S$  is real and so  $C = I$ . More generally, for the affine algebra  $X_r^{(1)}$  the conjugation  $C$  corresponds to a symmetry of the Dynkin diagram of  $X_r$ . For finite groups (Example 3),  $C$  takes  $(a, \chi)$  to  $(a^{-1}, \bar{\chi})$ . In RCFT,  $C$  is called *charge-conjugation*; it's a symmetry in quantum field theory which interchanges particles with their antiparticles (and so reverses charge, hence the name).

Because  $C$  is an involution, we know that the assignment (1.2b) defines a finite-dimensional representation of  $\mathrm{SL}_2(\mathbb{Z})$ , for any choice of modular data — hence the name. A surprising fact is that this representation usually (always?) seems to factor through a congruence subgroup. We'll return to this at the end of this section.

Perron-Frobenius theory, i.e. the spectral theory of nonnegative matrices (see e.g. [55]), has some immediate consequences. By **MD2** and our BASIC FACT, the Perron-Frobenius eigenvalue of  $N_a$  is  $\frac{S_{a0}}{S_{00}}$ ; hence we obtain the important inequality

$$S_{a0}S_{0b} \geq |S_{ab}| S_{00} . \quad (4.2a)$$

Unitarity of  $S$  applied to (4.2a) forces

$$\min_{a \in \Phi} S_{a0} = S_{00} . \quad (4.2b)$$

In other words the *q-dimensions*, defined to be the ratios  $\frac{S_{a0}}{S_{00}}$ , are bounded below by 1. The name ‘q-dimension’ comes from quantum groups (and also affine algebras (3.2c)), where one finds a q-deformed Weyl dimension formula. In RCFT,  $\frac{S_{a0}}{S_{00}} = \lim_{\tau \rightarrow 0+} \frac{\text{ch}_a(\tau)}{\text{ch}_0(\tau)}$ . In the subfactor picture (Example 6), the Jones index is the square of the q-dimension.

Cauchy-Schwarz and unitarity, together with (4.2a), gives us the curious inequality

$$\sum_{e \in \Phi} N_{ac}^e N_{bd}^e \leq \frac{S_{a0}}{S_{00}} \frac{S_{b0}}{S_{00}} \quad (4.2c)$$

for all  $a, b, c, d \in \Phi$ . So for instance  $N_{ab}^c \leq \min\{\frac{S_{a0}}{S_{00}}, \frac{S_{b0}}{S_{00}}, \frac{S_{c0}}{S_{00}}\}$ . Equality holds in (4.2c) only if  $S_{a0} = S_{b0} = S_{00}$  (i.e. only if  $a$  and  $b$  are *units* — see below). Other inequalities are possible, though perhaps not useful: e.g. Hölder gives us for all  $a \in \Phi$  and  $k, m = 1, 2, 3, \dots$  the following bounds on traces of powers of fusion matrices:

$$(\text{Tr}(N_a^k))^m \leq \|\Phi\|^{m-1} \text{Tr}(N_a^{km}) \quad (4.2d)$$

The inequality (4.2b) suggests that we look at those primaries  $a \in \Phi$  obeying the equality  $S_{a0} = S_{00}$ . Such primaries are called *simple-currents* in RCFT parlance (see e.g. [81,25] and references therein), but the much more obvious mathematical name is *units*. To any unit  $j \in \Phi$ , there is a phase  $\varphi_j : \Phi \rightarrow \mathbb{C}$  and a permutation  $J$  of  $\Phi$  such that  $j = J0$  and

$$S_{Ja,b} = \varphi_j(b) S_{a,b} \quad (4.3a)$$

$$T_{Ja,Ja} \overline{T_{aa}} = \overline{\varphi_j(a)} T_{jj} \overline{T_{00}} \quad (4.3b)$$

$$(T_{jj} \overline{T_{00}})^2 = \overline{\varphi_j(j)} \quad (4.3c)$$

Moreover, if  $J$  is order  $n$ , then  $\varphi_j(a)$  is an  $n$ th root of unity and  $T_{Ja,Ja} \overline{T_{aa}}$  is a  $2n$ th root of 1; when  $n$  is odd, the latter will in fact be an  $n$ th root of 1. To reflect the physics heritage, the permutation  $J$  corresponding to a unit  $j \in \Phi$  will be called a simple-current. The set of all simple-currents or units forms an abelian group (using composition of the permutations), called the *centre* of the modular data. Note that  $CJ = J^{-1}C$ , and  $N_{Ja,J'b}^{JJ'c} = N_{ab}^c$  for any simple-currents  $J, J'$ .

For instance, for a lattice  $\Lambda$ , all  $[a] \in \Phi$  are units, corresponding to permutation  $J_{[a]}([b]) = [a+b]$  and phase  $\varphi_{[a]}([b]) = e^{2\pi i a \cdot b}$ . For the affine algebra  $A_1^{(1)}$  at level  $k$  (recall

(3.5)), there is precisely one nontrivial unit, namely  $j = k$ , corresponding to  $J(a) = k - a$  and  $\varphi_j(a) = (-1)^a$ . More generally, to any affine algebra (except for  $E_8^{(1)}$  at  $k = 2$ ), the units correspond to symmetries of the extended Dynkin diagram. For  $A_1^{(1)}$  this symmetry interchanges the 0th and 1st nodes, i.e.  $J(\lambda_0\Lambda_0 + \lambda_1\Lambda_1) = \lambda_1\Lambda_0 + \lambda_0\Lambda_1$  (recall  $a = \lambda_1$ ); for  $A_r^{(1)}$  the centre is  $\mathbb{Z}/(r+1)\mathbb{Z}$ . In the finite group modular data, the units are the pairs  $(z, \psi)$  where  $z$  lies in the centre  $Z(G)$  of  $G$ , and  $\psi$  is a dimension-1 character of  $G$ . It corresponds to simple-current  $J_{(z,\psi)}(a, \chi) = (za, \psi\chi)$  and phase  $\varphi_{(z,\psi)}(a, \chi) = \overline{\psi(a)\chi(z)/\chi(e)}$ . The centre of this modular data will thus be isomorphic to the direct product  $Z(G) \times (G/G')$ , where  $G' = \langle ghg^{-1}h^{-1} \rangle$  is the commutator subgroup of  $G$ .

To see (4.3a), note first that (4.2a) tells us  $S_{0b} \geq |S_{jb}|$  for any unit  $j$ , and any  $b \in \Phi$ . However, unitarity then forces  $S_{0b} = |S_{jb}|$ , i.e. (4.3a) holds for  $a = 0$  (with  $J0$  defined to be  $j$ ), and some numbers  $\varphi_j(b)$  with modulus 1. Putting this into (2.1), we get  $N_j N_{Cj} = I$ , the identity matrix. But the only nonnegative integer matrices whose inverses are also nonnegative integer matrices, are the permutation matrices. This defines the permutation  $J$  of  $\Phi$ . Equation (4.3a) now follows from Cauchy-Schwartz applied to

$$1 = N_{j,a}^{Ja} = \sum_{d \in \Phi} \varphi_j(d) S_{ad} \overline{S_{Ja,d}}$$

The reason  $J \circ J' = J' \circ J$  is because the fusion matrices commute:  $N_{J \circ J'} = N_J N_{J'} = N_{J'} N_J = N_{J' \circ J}$ .

To see (4.3b), first write  $(ST)^3 = C$  as  $STS = \overline{T}S\overline{T}$ , then use that and (4.3a) to show  $(\overline{T}S\overline{T})_{Ja,0} = (\overline{T}S\overline{T})_{a,J0}$ . To see (4.3c), use (4.3b) with  $a = J^{-1}0$ , together with the fact that  $C$  commutes with  $T$ . Note that  $\varphi_j(j') = S_{j,j'}/S_{00} = \varphi_{j'}(j)$  and  $\varphi_{JJ'0}(a) = \varphi_j(a)\varphi_{j'}(a)$ , so  $\varphi_{J^k0}(a) = (\varphi_j(a))^k$ ; from all these and (4.3b) we get that

$$1 = T_{J^n0, J^n0} \overline{T_{00}} = \overline{\varphi_j(j)}^{n(n-1)/2} (T_{jj} \overline{T_{00}})^n$$

Equations (4.2a) and (4.3b) also follow from the curious equation

$$\overline{S_{ab}} T_{aa} T_{bb} \overline{T_{00}} = \sum_{c \in \Phi} N_{ab}^c T_{cc} S_{c0}$$

which is derived from (2.1) and  $STS = \overline{T}S\overline{T}$ .

Simple-currents and units play an important role in the theory of modular data and fusion rings. One place they appear is gradings. By a *grading* on  $\Phi$  we mean a map  $\varphi : \Phi \rightarrow \mathbb{C}^\times$  with the property that if  $N_{ab}^c \neq 0$  then  $\varphi(c) = \varphi(a)\varphi(b)$ . The phase  $\varphi_j$  coming from a unit is clearly a grading; a little more work [52] shows that any grading  $\varphi$  of  $\Phi$  corresponds to a unit  $j$  in this way. The multiplicative group of gradings, and the group of simple-currents (the centre), are naturally isomorphic.

Next, we will generalise the conjugation symmetry argument, to other Galois automorphisms. In particular, write  $\mathbb{Q}[S]$  for the field generated over  $\mathbb{Q}$  by all entries  $S_{ab}$ . Then for any Galois automorphism  $\sigma \in \text{Gal}(\mathbb{Q}[S]/\mathbb{Q})$ ,

$$\sigma(S_{ab}) = \epsilon_\sigma(a) S_{\sigma a, b} = \epsilon_\sigma(b) S_{a, \sigma b} \quad (4.4)$$

for some permutation  $c \mapsto \sigma c$  of  $\Phi$ , and some signs  $\epsilon_\sigma : \Phi \rightarrow \{\pm 1\}$ . Moreover, the complex numbers  $S_{ab}$  will necessarily lie in the cyclotomic extension  $\mathbb{Q}[\xi_n]$  of  $\mathbb{Q}$ , for some root of unity  $\xi_n := \exp[2\pi i/n]$ .

For a field extension  $\mathbb{K}$  of  $\mathbb{Q}$ ,  $\text{Gal}(\mathbb{K}/\mathbb{Q})$  denotes the automorphisms  $\sigma$  of  $\mathbb{K}$  fixing all rationals. Recall that each automorphism  $\sigma \in \text{Gal}(\mathbb{Q}[\xi_n]/\mathbb{Q})$  corresponds to an integer  $1 \leq \ell \leq n$  coprime to  $n$ , acting by  $\sigma(\xi_n) = \xi_n^\ell$ . Note that equation (4.4) tells us the power  $\sigma^{2\|\Phi\|!}$  will act trivially on each entry  $S_{ab}$ . In other words, the degree of the field extension  $[\mathbb{Q}[S] : \mathbb{Q}]$  is bounded by (in fact divides)  $2\|\Phi\|!$ . This is perhaps the closest we have to a finiteness result for modular data (see however [7] which obtains a bound for  $n$  in terms of  $\|\Phi\|$ , for the modular data arising in RCFT).

In other incarnations, this Galois action appears in the  $\chi(g) \mapsto \chi(g^\ell)$  symmetry of the character table of a finite group, and of the action of  $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$  on level  $N$  modular functions. Equation (4.4) was first shown in [20] and a related symmetry for commutative association schemes was found in [74]. The analogue of cyclotomy isn't known for association schemes. The reason is the additional 'self-duality' property of the fusion ring, i.e. the fact that  $S = S^t$  or more generally (2.4).

Recall from Section II that a fusion ring  $R = \mathcal{F}(\Phi, N)$  is isomorphic to a direct sum of number fields. The Galois orbits determine these fields. In particular, for any Galois orbit  $[d]$  in  $\Phi$ , let  $\mathbb{K}_{[d]}$  denote the field generated by all numbers of the form  $\frac{S_{ab}}{S_{0b}}$  for  $a \in \Phi$  and  $b \in [d]$ . Then  $R$  is isomorphic as a  $\mathbb{Q}$ -algebra to the direct sum  $\bigoplus_{[d]} \mathbb{K}_{[d]}$ . We gave the  $A_1^{(1)}$  level  $k$  example in Section II.

The Galois action for the lattice modular data is simple: the Galois automorphisms  $\sigma = \sigma_\ell$  correspond to integers  $\ell$  coprime to the determinant  $|\Lambda|$ ;  $\sigma_\ell$  takes  $[a]$  to  $[\ell a]$ , and all parities  $\epsilon_\ell([a]) = +1$ . The Galois action for the affine algebras is quite interesting (see e.g. [1]), and can be expressed geometrically using the action of the affine Weyl group on the weight lattice of  $X_r$ . Both  $\epsilon_\ell(\lambda) = \pm 1$  will occur. For finite groups,  $\sigma_\ell$  takes  $(a, \chi)$  to  $(a^\ell, \sigma_\ell \circ \chi)$ , and again all  $\epsilon_\ell(a, \chi) = +1$ .

The presence of the Galois action (4.4) is an effective criterion (necessary and sufficient) on the matrix  $S$  for the numbers in (2.1) to be rational. It would be very desirable to find effective conditions on  $S$  such that the fusion coefficients are nonnegative, or integral. At present the best results along these lines are, respectively, the inequalities (4.2), and the fact that the ratios  $\frac{S_{ab}}{S_{0b}}$  are algebraic integers (since they are eigenvalues of integer matrices). When there are units, then (4.3a) provides an additional strong constraint on nonnegativity.

Whenever a structure is studied, of fundamental importance are the structure-preserving maps. It is through these maps that different examples of the structure can be compared. By a *fusion-homomorphism*  $\pi$  between fusion rings  $\mathcal{F}(\Phi, N)$  and  $\mathcal{F}(\Phi', N')$  we mean a ring homomorphism for which  $\pi(\Phi) \subseteq \Phi'$ . *Fusion-isomorphisms* and *fusion-automorphisms* are defined in the obvious ways. All fusion-isomorphisms between affine algebra fusion rings are known. Most of them are in fact fusion-automorphisms, and are constructed in simple ways from the symmetries of the Dynkin diagrams. Here are some basic general facts about fusion-homomorphisms:

**PROPOSITION 3.** *Let  $\pi : \Phi \rightarrow \Phi'$  be a fusion-homomorphism between any two fusion rings. Then*

- (a)  $\pi 0 = 0'$  and  $\pi(a^*) = \pi(a)^*$ , and  $\pi$  takes units of  $\Phi$  to units of  $\Phi'$ .
- (b) There exists a map  $\pi' : \Phi' \rightarrow \Phi$  such that

$$\frac{S'_{\pi a, b'}}{S'_{0', b'}} = \frac{S_{a, \pi' b'}}{S_{0, \pi' b'}} \quad \forall a \in \Phi, b' \in \Phi'$$

- (c) If  $\pi a = \pi b$ , then  $b = Ja$  for some simple-current  $J$ . In addition, this  $J$  will obey  $\pi(Jd) = \pi(d)$  for all  $d \in \Phi$ , and (provided  $J$  is nontrivial) there can be no  $J$ -fixed-points in  $\Phi$ .
- (d) If  $\pi$  is surjective, then  $\pi' : \Phi' \rightarrow \Phi$  is an injective fusion-homomorphism, and

$$S'_{\pi a, b'} = \sqrt{\ker(\pi)} S_{a, \pi' b'}$$

Part (a) follows from **F1** and **F3**. Part (b) follows because  $\frac{S'_{\pi a, b'}}{S'_{0', b'}}$  is a 1-dimensional representation of the  $\Phi'$  fusion ring. To get (c), consider  $(\pi a)(\pi b)^* = \pi(ab^*)$ . If  $f$  is a fixed-point of  $J$  in (c), count the multiplicity of the identity  $0'$  in the fusion product  $(\pi f) \cdot (\pi f)^*$ . To see (d), apply (c) to

$$\sum_a \left| \frac{S'_{\pi a, b'}}{S'_{0', b'}} \right|^2 = \sum_a \left| \frac{S_{a, \pi' b'}}{S_{0, \pi' b'}} \right|^2$$

For example, fix any units  $j, j' \in \Phi$  of equal order  $n$ . Then  $a \mapsto J^{Q'(a)} a$  defines a fusion-endomorphism, where we write  $\varphi_{j'}(a) = \exp[2\pi i Q'(a)/n]$ . It will be a fusion-automorphism iff  $Q'(j) + 1$  is coprime to  $n$ . For another example, take any Galois automorphism  $\sigma$  for which  $\sigma(S_{00}^2) = S_{00}^2$ , or equivalently  $\sigma 0 = J 0$  for some simple-current  $J$ . Then  $a \mapsto J \sigma a$  is a fusion-automorphism. For this Galois example  $\pi' = \pi$ , while for the simple-current one  $\pi'(b) = J'^{Q(b)} b$ .

The map  $\pi'$  of Prop. 3(b) won't in general be a fusion-homomorphism. E.g. consider the fusion-homomorphism  $\pi : \{[0], [1]\} \rightarrow \{0, 1, \dots, k\}$  between the fusion ring of the lattice  $\Lambda = \sqrt{2}\mathbb{Z}$  and the fusion ring for  $A_1^{(1)}$  level  $k$ , given by  $\pi([0]) = 0, \pi([1]) = k$ . Then  $\pi'$  is given by  $\pi'(a) = [a]$ .

A very desirable property for modular data to possess is:

**CONGRUENCE SUBGROUP PROPERTY.** [21] Let  $N$  be the order of the matrix  $T$ , so  $T^N = I$ , and let  $\rho$  be the representation of  $SL_2(\mathbb{Z})$  coming from the assignment (1.2b). Then  $\rho$  factors through the congruence subgroup

$$\Gamma(N) := \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

and so (1.2b) in fact defines a representation of the finite group  $SL_2(\mathbb{Z}/N\mathbb{Z})$ . Moreover, the characters (1.1) are modular functions for  $\Gamma(N)$ . The entries  $S_{ab}$  all lie in the cyclotomic field  $\mathbb{Q}[\exp(2\pi i/N)]$ , and for any Galois automorphism  $\sigma_\ell$ ,

$$T_{\sigma_\ell a, \sigma_\ell a} = T_{aa}^{\ell^2} \quad \forall a \in \Phi \tag{4.5}$$

For example, the modular data from Examples 1–3 in Section III all obey this property. In particular, affine algebra characters  $\chi_\lambda$  are essentially lattice theta functions. It would be valuable to find examples of modular data which do *not* obey this property. For much more discussion, see [21,7]. In those papers, considerable progress was made towards clarifying its role (and existence) in modular data. For example:

**PROPOSITION 4.** [21] *Consider any modular data. Let  $N$  be the order of  $T$ , and suppose that  $N$  is either coprime to  $p = 2$  or  $p = 3$ . Then the corresponding  $SL_2(\mathbb{Z})$  representation factors through  $\Gamma(N)$ , provided (4.5) holds for  $\ell = p$ .*

In the remaining case, i.e. when 6 divides  $N$ , more conditions are needed; these are also given in [21]. It is tempting to think that this is a good approach to verifying that rational VOA characters are modular functions. It also leads, via [32], to a promising approach to classifying modular data.

Assuming some additional structure from RCFT, [7] recently established the congruence property ([21] had previously proved the  $\Gamma(N)$  part when  $T$  has odd order). Though this is clearly an impressive feat, what it means in the more general context of modular data isn't clear: it is difficult to explicitly write down the additional axioms needed to supplement our definition of modular data, in order that the necessary calculations go through.

## V. Modular Invariants and NIM-reps

A *modular invariant* is a matrix  $M$ , rows and columns labeled by  $\Phi$ , obeying:

- MI1.**  $MS = SM$  and  $MT = TM$ ;
- MI2.**  $M_{ab} \in \mathbb{Z}_{\geq}$  for all  $a, b \in \Phi$ ; and
- MI3.**  $M_{00} = 1$ .

As usual we write  $\mathbb{Z}_{\geq}$  for the nonnegative integers. The simplest example of a modular invariant is of course the identity matrix  $M = I$ . Another example is conjugation  $C$ . All of the modular invariants for  $A_1^{(1)}$  at level  $k$  are given below in (6.1).

Why are modular invariants interesting? Most importantly, they are central to the task of classifying RCFTs. The genus-1 ‘vacuum-to-vacuum amplitude’ (=partition function)  $\mathcal{Z}(\tau)$  of the theory looks like (1.3c). It assigns to the torus  $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$  the complex number  $\mathcal{Z}(\tau)$ . But the moduli space of conformally equivalent tori is the orbit space  $SL_2(\mathbb{Z})\backslash\mathbb{H}$ , where the action is given by  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau+b}{c\tau+d}$ . Thus the partition function  $\mathcal{Z}(\tau)$  must be invariant under this natural action of the modular group  $SL_2(\mathbb{Z})$ , which gives us **MI1**. The coefficients  $M_{ab}$  count the primary fields  $|\phi_a, \phi_b\rangle$  in the state space  $\mathcal{H}$ , i.e. the number of times the module  $A_a \otimes A_b$  of left chiral algebra  $\times$  right chiral algebra, appears in  $\mathcal{H}$ . That gives us **MI2**. And the uniqueness of the vacuum  $|0, 0\rangle$  means **MI3**. That is to say, the coefficient matrix  $M$  of an RCFT partition function is a modular invariant. It is believed that an RCFT is uniquely specified by the knowledge of its partition function, its (left and right) chiral algebras (=VOAs), and the so-called structure constants. In any case, an important fingerprint of the RCFT is its partition function  $\mathcal{Z}$ , i.e. its modular invariant  $M$ .

Another motivation for studying modular invariants is the extensions  $\mathcal{V} \subset \mathcal{V}'$  of rational VOAs (similar remarks hold for braided subfactors). Let  $M_i$  and  $M'_j$  be the irreducible modules of  $\mathcal{V}$  and  $\mathcal{V}'$ , respectively. Then each  $M'_j$  will be a  $\mathcal{V}$ -module. A rational VOA should have the complete reducibility property, so each  $M'_j$  should be expressible as a direct sum of  $M_i$ 's — these are called the branching rules. As mentioned in Example 5, we would expect that the characters (1.1) of a rational VOA should yield (some form of) modular data via (1.2a). So the diagonal sum  $\sum_j |\text{ch}_{M'_j}|^2$  should be invariant under the  $\text{SL}_2(\mathbb{Z})$ -action; rewriting the  $\text{ch}'_{M'_j}$ 's in terms of the  $\text{ch}_{M_i}$ 's via the branching rules yields a modular invariant for  $\mathcal{V}$ .

For instance, the VOA  $L(\Lambda_0)'$  corresponding to the affine algebra  $G_2^{(1)}$  level 1 contains the VOA  $L(28\Lambda_0)$  corresponding to  $A_1^{(1)}$  at level 28. We get the branching rules  $L(\Lambda_0)' = L(0) \oplus L(10) \oplus L(18) \oplus L(28)$  and  $L(\Lambda_2)' = L(6) \oplus L(12) \oplus L(16) \oplus L(22)$ , where  $L(\lambda_1) := L(\lambda)$ . This corresponds to the  $A_1^{(1)}$  level 28 modular invariant given below in (6.1f).

So knowing the modular invariants for some VOA  $\mathcal{V}$  gives considerable information concerning its possible ‘nice’ extensions  $\mathcal{V}'$ . For instance, we are learning that the only finite ‘rational’ extensions of a generic affine VOA are those studied in [29] (‘simple-current extensions’) and whose modular data is conjecturally given in [41].

Another reason for studying modular invariants is that the answers are often surprising. Lists arising in math from complete classifications tend to be about as stale as phonebooks, but to give some samples:

- the  $A_1^{(1)}$  modular invariants fall into the A-D-E metapattern;
- the  $A_2^{(1)}$  modular invariants have connections with Jacobians of Fermat curves; and
- the  $(U(1) \oplus \dots \oplus U(1))^{(1)}$  modular invariants correspond to rational points on Grassmannians.

We will discuss this point a little more next section. These ‘coincidences’, presumably, have something to do with the nontrivial connections between RCFT and several areas of math, but it also is due to the beauty of the combinatorics of Lie characters evaluated at elements of finite order (3.2c).

In any case, in this section we will study the modular invariants corresponding to a given choice of modular data. For lattices, the classification is easy (use (5.2) below). For many finite groups, the classification typically will be hopeless — e.g. the alternating group  $A_5$ , which has only 22 primaries, has a remarkably high number (8719) of modular invariants [5]. For affine algebra modular data, the classification of modular invariants seems to be just barely possible, and the answer is that (generically) the only modular invariants are constructed in straightforward ways from symmetries of the Dynkin diagrams.

Commutation with  $T$  is trivial to solve, since  $T$  is diagonal: it yields the selection rule

$$M_{ab} \neq 0 \Rightarrow T_{aa} = T_{bb} \quad (5.1)$$

This isn’t as useful as it looks; commutation with  $S$  (or equivalently, the equation  $SMS = M$ ) is more subtle, but far more valuable.

An immediate observation is that there are only finitely many modular invariants

associated to given modular data. This follows for instance from

$$1 = M_{00} = \sum_{a,b \in \Phi} S_{0a} M_{ab} S_{b0} \geq S_{00}^2 \sum_{a,b \in \Phi} M_{ab}$$

We will find that each basic symmetry of the  $S$  matrix yields a symmetry of the modular invariants, a selection rule telling us that certain entries of  $M$  must vanish, and a way to construct new modular invariants.

First consider simple-currents  $J, J'$ . Equation (4.3a) and positivity tell us

$$M_{J0,J'0} = \left| \sum_{c,d \in \Phi} \varphi_J(c) S_{0c} M_{cd} \overline{S_{d0}} \overline{\varphi_{J'}(d)} \right| \leq \sum_{c,d} S_{0c} M_{cd} S_{d0} = M_{00} = 1 \quad (5.2a)$$

Thus  $M_{J0,J'0} \neq 0$  implies  $M_{J0,J'0} = 1$ , as well as the selection rule

$$M_{cd} \neq 0 \Rightarrow \varphi_J(c) = \varphi_{J'}(d) \quad (5.2b)$$

A similar calculation yields the symmetry

$$M_{J0,J'0} \neq 0 \Rightarrow M_{Ja,J'b} = M_{ab} \quad \forall a, b \in \Phi \quad (5.2c)$$

The most useful application of simple-currents to modular invariants is to their construction. In particular, let  $J$  be a simple-current of order  $n$ . Then we learned in (4.3) that  $\varphi_j(a)$  is an  $n$ th root of 1, and that  $(T_{jj} \overline{T_{00}})^{2n} = 1$  and in fact  $(T_{jj} \overline{T_{00}})^n = 1$  when  $n$  is odd. That is to say, we can find integers  $r_j$  and  $Q_j(a)$  obeying

$$\varphi_j(a) = \exp[2\pi i \frac{Q_j(a)}{n}] , \quad T_{jj} \overline{T_{00}} = \exp[\pi i r_j \frac{n-1}{n}]$$

For  $n$  odd, choose  $r_j$  to be even (by adding  $n$  to it if necessary). Now define the matrix  $\mathcal{M}[J]$  by [81]

$$\mathcal{M}[J]_{ab} = \sum_{\ell=1}^n \delta_{J^\ell a, b} \delta\left(\frac{Q_j(a)}{n} + \frac{\ell}{2n} r_j\right) \quad (5.3)$$

where  $\delta(x) = 1$  when  $x \in \mathbb{Z}$  and is 0 otherwise. This matrix  $\mathcal{M}[J]$  will be a modular invariant iff  $(T_{jj} \overline{T_{00}})^n = 1$  (i.e. iff  $r_j$  is even), and a permutation matrix iff  $T_{jj} \overline{T_{00}}$  is a primitive  $n$ th root of 1. When  $n$  is even, (4.3c) says  $(T_{jj} \overline{T_{00}})^n = 1$  iff  $\varphi_j(j)^{n/2} = 1$ .

For instance, taking  $J = id$  we get  $\mathcal{M}[id] = I$ . The affine algebra  $A_1^{(1)}$  at level  $k$  has a simple-current with  $r_j = k$  given by  $Ja = k - a$ ; for even  $k$  the matrix  $\mathcal{M}[J]$  is the modular invariant called  $\mathcal{D}_{\frac{k}{2}+2}$  below in (6.1b),(6.1c).

Now look at the consequences of Galois. Applying the Galois automorphism  $\sigma$  to  $M = SM\overline{S}$  yields from (4.4) and  $M_{ab} \in \mathbb{Q}$  the fundamental equation

$$M_{ab} = \sum_{c,d \in \Phi} \epsilon_\sigma(a) S_{\sigma a, c} M_{cd} \overline{S_{d, \sigma b}} \epsilon_\sigma(b) = \epsilon_\sigma(a) \epsilon_\sigma(b) M_{\sigma a, \sigma b} \quad (5.4a)$$

Because  $M_{ab} \geq 0$ , we obtain the selection rule

$$M_{ab} \neq 0 \Rightarrow \epsilon_\sigma(a) = \epsilon_\sigma(b) \quad \forall \sigma \quad (5.4b)$$

and the symmetry

$$M_{\sigma a, \sigma b} = M_{ab} \quad \forall \sigma \quad (5.4c)$$

Of all the equations (5.2) and (5.4), (5.4b) is the most valuable. A way to construct modular invariants from Galois was first given in [40] but isn't useful for constructing affine algebra modular invariants and so won't be repeated here.

There are other very useful facts, which space prevents us from describing. For instance, we have the inequality

$$\sum_{b \in \Phi} S_{ab} M_{b0} \geq 0 \quad (5.5)$$

Perron-Frobenius tells us many things, e.g. that any modular invariant  $M$  obeying  $M_{0a} = \delta_{0a}$  must be a permutation matrix. For affine algebra modular invariants, the Lie theory of the underlying finite-dimensional Lie algebra plays a crucial role, thanks largely to (3.2c).

Closely related to modular invariants is the notion of *NIM-rep* (short for ‘nonnegative integer representation’ [12]) or equivalently *fusion graph*. These originally arose in two *a priori* unrelated contexts: the analysis, starting with Cardy's fundamental paper [16], of boundary RCFT; and Di Francesco–Zuber's largely empirical attempt [24] to understand and generalise the A-D-E metapattern appearing in  $A^{(1)}$  modular invariants, by attaching graphs to each conformal field theory.

A *NIM-rep*  $\mathcal{N}$  is a nonnegative integer representation of the fusion ring, that is, an assignment  $a \mapsto \mathcal{N}_a$  to each  $a \in \Phi$  of a matrix  $\mathcal{N}_a$  with nonnegative integer entries, obeying  $\mathcal{N}_a \mathcal{N}_b = \sum_c N_{ab}^c \mathcal{N}_c$ . In addition we require that  $\mathcal{N}_0 = I$  and that transpose and conjugation be related by  $\mathcal{N}_a^t = \mathcal{N}_{Ca}$ , for all  $a \in \Phi$ .

Two obvious examples of NIM-reps are the fusion matrices,  $a \mapsto N_a$ , and their transposes  $a \mapsto N_a^t$ . The rows and columns of most NIM-reps however won't be labelled by  $\Phi$ , in fact we will see that the dimension of the NIM-rep should equal the trace  $\text{Tr}(M)$  of some modular invariant.

Just as it is convenient to replace a Cartan matrix by its Dynkin diagram, so too is it convenient to realise  $\mathcal{N}_a$  by a (directed multi-)graph: we put a node for each row/column, and draw  $(\mathcal{N}_a)_{\alpha\beta}$  edges directed from  $\alpha$  to  $\beta$ . We replace each pair of arrows  $\alpha \rightarrow \beta, \beta \rightarrow \alpha$ , with a single undirected edge connecting  $\alpha$  and  $\beta$ . These graphs are called *fusion graphs*, and are often quite striking.

NIM-reps correspond in RCFT to the 1-loop vacuum-to-vacuum amplitude  $\mathcal{Z}_{\alpha\beta}(t)$  of an open string, or equivalently of a cylinder whose edge circles are labelled by ‘conformally invariant boundary states’  $|\alpha\rangle, |\beta\rangle$  [16,80,42,11]. In string theory these are called the ‘Chan-Paton degrees-of-freedom’ and are placed at the endpoints of open strings. The real variable  $-\infty < t < \infty$  here is the modular parameter for the cylinder, and plays the same role here that  $\tau \in \mathbb{H}$  plays in  $\mathcal{Z}(\tau)$ . In particular we get (1.4), where the matrices  $(\mathcal{N}_a)_{\alpha\beta} = \mathcal{N}_{a\alpha}^\beta$  define a NIM-rep. These (finitely many) boundary states  $\alpha$  are the indices for the rows and columns of each matrix  $\mathcal{N}_a$ .

By the usual arguments (see Section IV) we can simultaneously diagonalise all  $\mathcal{N}_a$ , and the eigenvalues of  $\mathcal{N}_a$  will be  $S_{ab}/S_{0b}$  for  $b$  in some multi-set  $\mathcal{E} = \mathcal{E}(\mathcal{N})$  (i.e. the elements of  $\mathcal{E}$  come with multiplicities). This multi-set  $\mathcal{E}$  depends only on  $\mathcal{N}$  (i.e. is independent of  $a \in \Phi$ ), and is called the *exponents* of the NIM-rep.

Two NIM-reps  $\mathcal{N}, \mathcal{N}'$  are regarded as equivalent if there is a simultaneous permutation  $\pi$  of the rows and columns such that  $\pi\mathcal{N}_a\pi^{-1} = \mathcal{N}'_a$  for all  $a \in \Phi$ . For example, the two NIM-reps given earlier are equivalent:  $N_a^t = CN_aC^{-1}$ . We write  $\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}''$ , and call  $\mathcal{N}$  *reducible*, if the matrices  $\mathcal{N}_a$  can be simultaneously written as direct sums  $\mathcal{N}_a = \mathcal{N}'_a \oplus \mathcal{N}''_a$ . Necessarily, the summands  $\mathcal{N}'$  and  $\mathcal{N}''$  themselves will be NIM-reps. Irreducibility is equivalent to demanding that the identity 0 occurs in  $\mathcal{E}(\mathcal{N})$  with multiplicity 1. We are interested in irreducible equivalence classes of NIM-reps — there will be only finitely many [53].

Two useful facts are: the Perron-Frobenius eigenvalue of  $\mathcal{N}_a$  is the q-dimension  $\frac{S_{a0}}{S_{00}}$  (we'll see this used next section); and for all  $a \in \Phi$ ,

$$\sum_{b \in \mathcal{E}} \frac{S_{ab}}{S_{0b}} = \text{Tr}(\mathcal{N}_a) \in \mathbb{Z}_{\geq} \quad (5.6)$$

The consequences of the simple-current and Galois symmetries are also important and are worked out in [53].

By the *exponents* of a modular invariant  $M$  we mean the multi-set  $\mathcal{E}_M$  where  $a \in \Phi$  appears with multiplicity  $M_{aa}$ . RCFT [16,11] is thought to require that each modular invariant  $M$  have a companion NIM-rep  $\mathcal{N}$  with the property that

$$\mathcal{E}_M = \mathcal{E}(\mathcal{N}) \quad (5.7)$$

So the size of the matrices  $\mathcal{N}_a$ , i.e. the dimension of the NIM-rep, should equal the trace  $\text{Tr}(M)$  of the modular invariant. For instance, the fusion matrix NIM-rep  $a \mapsto N_a$  corresponds to the modular invariant  $M = I$ . However, there doesn't seem to be a general expression for the NIM-rep (if it exists) of the next simplest modular invariant, the conjugation  $M = C$ .

Incidentally, the inequality (5.6) is automatically obeyed by the exponents  $\mathcal{E} = \mathcal{E}_M$  of any modular invariant  $M$ :

$$\sum_{b \in \mathcal{E}_M} \frac{S_{ab}}{S_{0b}} = \text{Tr}(MD_a) = \text{Tr}(\overline{S}SM D_a) = \text{Tr}(MSD_a \overline{S}) = \text{Tr}(MN_a) \in \mathbb{Z}_{\geq}$$

Note that the NIM-rep definition depends on  $S$ , while a modular invariant also sees  $T$ . One consequence of this is the following. Suppose there is a primary  $a \in \Phi$  such that

$$T_{bb} = T_{cc} \Rightarrow S_{ab} \overline{S_{ac}} \geq 0 \quad \forall b, c \in \Phi \quad (5.8)$$

Then  $M_{aa} = \sum_{b,c} S_{ab} M_{bc} \overline{S_{ac}} > 0$  and so  $a \in \mathcal{E}_M$ . It is thus natural to require of a NIM-rep  $\mathcal{N}$  that any such primary  $a \in \Phi$  must appear in  $\mathcal{E}(\mathcal{N})$  with multiplicity  $\geq 1$ , because otherwise no modular invariant  $M$  could be found obeying (5.7).

It should be mentioned that, from the RCFT point of view, the constraint (5.7) is not as ‘carved in stone’ as **MI1–MI3**. Our treatment here of NIM-reps reflects the current understanding, but it is still based on unproven physical assumptions (‘completeness of boundary conditions’) and perhaps in the future will require some modification. Also, we’re ignoring here the ‘pairing’=‘gluing automorphism’  $\omega$  of e.g. [42], although here this isn’t a serious omission. But an independent justification for studying NIM-reps, and a strong hint that this RCFT picture is not too naive, comes from subfactors.

NIM-reps and modular invariants appear very naturally in the subfactor picture (Example 6) [33,76,12], again paired by the relation (5.7). In this remarkable picture it is possible to interpret not only the diagonal entries of the modular invariant, but in fact all entries [75,12] (this was already anticipated in [24]). Extend the setting of Example 6 by considering a braided system of endomorphisms for a type III subfactor  $N \subset M$ . Here, the primaries  $\Phi = {}_{N\chi_N}$  consist of irreducible endomorphisms of  $N$ , while the rows and columns of our NIM-rep will be indexed by irreducible homomorphisms  $a \in {}_{M\chi_M}$ ,  $a : N \rightarrow M$ . The fusion-like ring of  ${}_{N\chi_N}$  will be commutative, i.e. be a true fusion ring; that of  ${}_{M\chi_M}$  however will generally be noncommutative. There is a simple expression [12] for the corresponding modular invariant using ‘ $\alpha$ -induction’ (a process of inducing an endomorphism from  $N$  to  $M$  using the braiding  $\epsilon^\pm$ ): we get  $M_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$  where the dimension  $\langle , \rangle$  is defined in Example 6. Then the (complexified) fusion algebra of  ${}_{M\chi_M}$  will be isomorphic (as a complex algebra) to  $\oplus_{\lambda,\mu} \text{GL}_{M_{\lambda\mu}}(\mathbb{C})$ . The NIM-rep is essentially  $\alpha$ :  $(\mathcal{N}_\lambda)_{a,b} = \langle b, \alpha_\lambda^\pm a \rangle$  (either choice of  $\alpha^\pm$  gives the same matrix) [12]. This NIM-rep arises as a natural action of  ${}_{M\chi_M}$  on  ${}_{N\chi_N}$ . As these partition functions of tori and cylinders appear so nicely here, it is tempting to ask about other surfaces, especially the Möbius band and Klein bottle, which also play a basic role in boundary RCFT [80].

We won’t speak much more here about NIM-reps — see e.g. [24,11,53] and references therein for more of the theory and classifications (and graphs!). Typically, what has happened in the classifications thus far is that there are slightly more NIM-reps than modular invariants, but their classifications match surprisingly well. For instance, the irreducible NIM-reps of  $A_1^{(1)}$  have  $\mathcal{N}_1$  equal to the incidence matrix of the A-D-E graphs and tadpoles [24] — compare with the list of modular invariants for  $A_1^{(1)}$  in (6.1)! However there are places where many modular invariants lack a corresponding NIM-rep (this happens for instance for the orthogonal algebras at level 2 [53]). The simplest examples of modular invariants lacking NIM-reps occur for  $B_4^{(1)}$  level 2, and the symmetric group  $S_3$ .

A tempting guess is that almost all of the enormous numbers of modular invariants associated to finite group modular data will likewise fail to have a corresponding NIM-rep. Recall that the Galois parities  $\epsilon_\ell$  for the finite group modular data are all +1, and hence the constraint (5.4b) becomes trivial. As a general rule, the number of modular invariants is inversely related to the severity (5.4b) possesses for that choice of modular data.

The moral of the story seems to be the following. The definition of modular invariants didn’t come to us from God; it came to us from men like Witten, Cardy, ... The surprising thing is that so often their classification yields interesting answers. A modular invariant may not correspond to a CFT (we have infinitely many examples where it fails to), and the modular invariant may correspond to different CFTs (though all known examples of this are artificial, due to our characters depending on too few variables to distinguish the

representations of the maximally extended VOAs). But — at least for most affine algebras and levels — it seems they’re *usually* in one-to-one correspondence.

In any case, classifying modular invariants, and comparing their lists to those of NIM-reps, is a natural task and has led to interesting findings (see e.g. the review [93]).

## VI. Affine Algebra Modular Invariant Classifications

The most famous modular invariant classification was the first. In (3.5) we gave explicitly the modular data for the affine algebra  $A_1^{(1)}$  at level  $k$ . Its complete list of modular invariants is [15] (using the simple-current  $J_a = k - a$ )

$$\mathcal{A}_{k+1} = \sum_{a=0}^k |\chi_a|^2 , \quad \forall k \geq 1 \quad (6.1a)$$

$$\mathcal{D}_{\frac{k}{2}+2} = \sum_{a=0}^k \chi_a \overline{\chi_{J^a a}} , \quad \text{whenever } \frac{k}{2} \text{ is odd} \quad (6.1b)$$

$$\mathcal{D}_{\frac{k}{2}+2} = |\chi_0 + \chi_{J0}|^2 + |\chi_2 + \chi_{J2}|^2 + \cdots + 2|\chi_{\frac{k}{2}}|^2 , \quad \text{whenever } \frac{k}{2} \text{ is even} \quad (6.1c)$$

$$\mathcal{E}_6 = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2 , \quad \text{for } k = 10 \quad (6.1d)$$

$$\begin{aligned} \mathcal{E}_7 = & |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 \\ & + \chi_8 (\overline{\chi_2 + \chi_{14}}) + (\chi_2 + \chi_{14}) \overline{\chi_8} + |\chi_8|^2 , \quad \text{for } k = 16 \end{aligned} \quad (6.1e)$$

$$\mathcal{E}_8 = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2 , \quad \text{for } k = 28 \quad (6.1f)$$

Each of these is identified with a (finite) Dynkin diagram, in such a way that the Coxeter number  $h$  of the diagram equals  $k + 2$ , and the *exponents* of the corresponding Lie algebra are given by  $1 + \mathcal{E}_M$  (recall the definition of exponents  $\mathcal{E}_M$  of a modular invariant, given at the end of last section). The exponents of the Lie algebra are the numbers  $m_i$ , where  $4 \sin^2(\pi \frac{m_i}{h})$  are the eigenvalues of the Cartan matrix. For instance, the Dynkin diagram  $D_8$  has Coxeter number 14 and exponents 1, 3, 5, 7, 7, while  $\mathcal{D}_8$  occurs at level 12 and has exponents  $\mathcal{E} = \{0, 2, 4, 6, 6\}$ .

The A-D-E pattern appears in many places in math and mathematical physics [60]: besides the simply-laced Lie algebras and  $A_1^{(1)}$  modular invariants, these diagrams also classify simple singularities, finite subgroups of  $SU_2(\mathbb{C})$ , subfactors with Jones index  $< 4$ , representations of quivers, etc. There seem to be two more-or-less inequivalent A-D-E patterns, one corresponding to the finite A-D-E diagrams, and the other corresponding to the affine (=extended) A-D-E diagrams. For instance, the modular invariants identify with the finite ones, while the finite subgroups of  $SU_2(\mathbb{C})$  match with the affine ones. This suggests that a direct relation between e.g. the modular invariants and those finite subgroups could be a little forced. Patterns such as A-D-E are usually explained by identifying an underlying combinatorial fact which is responsible for its various incarnations. The A-D-E combinatorial fact is probably the classification of symmetric matrices over  $\mathbb{Z}_{\geq}$ , with no diagonal entries, and with maximal eigenvalue  $< 2$  (for the finite diagrams) and  $= 2$  (for the affine ones). Perhaps the only A-D-E classification which still resists this ‘explanation’ is that of  $A_1^{(1)}$  modular invariants. This is in spite of considerable effort (and some

progress) by many people. The present state of affairs, and also a much simpler proof on the lines sketched in the previous section, is provided by [51].

Many other classes of affine algebras and levels have been classified. The main ones are:  $A_2^{(1)}$ ,  $(A_1 + A_1)^{(1)}$ , and  $(U(1) + \cdots + U(1))^{(1)}$ , for all levels  $k$ ; and  $A_r^{(1)}, B_r^{(1)}, D_r^{(1)}$  for all ranks  $r$ , but with levels restricted to  $k \leq 3$ . See e.g. [50] for references to these results.

Has A-D-E been spotted in these other lists? No. However, a remarkable connection [79] has been observed between the  $A_2^{(1)}$  level  $k$  modular invariants, and the Jacobian of the Fermat curve  $x^{k+3} + y^{k+3} + z^{k+3} = 0$ . In particular, the  $A_2^{(1)}$  Galois selection rule (5.4b) and the analysis of the simple factors in the Jacobian are essentially the same. This link between Fermat and  $A_2^{(1)}$  is still unexplained, and how it extends to the other algebras, e.g. perhaps  $A_r^{(1)}$  level  $k$  relates to  $x_1^{k+r+1} + x_2^{k+r+1} + \cdots + x_{r+1}^{k+r+1} = 0$ ?, is still unclear. However, Batyrev [8] has suggested some possibilities involving toric geometry.

The third ‘sample’ listed last section (relating  $(U(1) \oplus \cdots \oplus U(1))^{(1)}$  modular invariants to the Grassmannians) suggests a different link with geometry. The Grassmannian is essentially the moduli space of Narain compactifications of a (classical) string theory, so perhaps other families of modular invariants can be regarded as special points on other finite-dimensional moduli spaces.

Though there are no other appearances of A-D-E, there is a rather natural way to assign (multi-di)graphs to modular invariants, generalising the A-D-E pattern for  $A_1^{(1)}$ . Note first that we can classify the  $A_1^{(1)}$  NIM-reps [24]:  $\mathcal{N}_1$  must be symmetric and have Perron-Frobenius eigenvalue  $\frac{S_{10}}{S_{00}} = 2 \cos(\frac{\pi}{k+2}) < 2$ ; thus the graph associated to  $\mathcal{N}_1$  must be an A-D-E Dynkin diagram, or a tadpole. The tadpoles can be discarded, since they don’t correspond via (5.7) to a modular invariant. Given  $\mathcal{N}_1$ , all other  $\mathcal{N}_a$  can be recursively obtained using the special case  $\mathcal{N}_1 \mathcal{N}_i = \mathcal{N}_{i+1} + \mathcal{N}_{i-1}$  of (3.5c). The result is a NIMrep.

In this way, we find that the Dynkin diagram which (6.1) assigned to a given  $A_1^{(1)}$  modular invariant  $M$  is precisely the graph whose adjacency matrix equals the generator  $\mathcal{N}_1$  of the unique NIMrep compatible with  $M$  in the sense of (5.7). Likewise, we should assign to the modular invariants of e.g.  $A_2^{(1)}$  the multi-digraph  $\mathcal{N}_{\Lambda_1}$  generating the corresponding NIMrep. The NIM-reps for  $A_2^{(1)}$  are not yet classified, but at least one has been found for each  $M$  [24, 76, 11, 12].

There is a simple reason why the tadpole can’t correspond to an  $A_1^{(1)}$  modular invariant. Note that the unit  $a = k$  satisfies (5.8), and thus will lie in any  $\mathcal{E}_M$ . However,  $k$  is not an exponent of the tadpole, and thus there can be no solution  $M$  in (5.7) for the choice  $\mathcal{N} = \text{tadpole}$ . More generally, this suggests refining the definition of NIMrep: many extraneous (unphysical?) NIM-reps can be avoided, by requiring  $a \in \mathcal{E}(\mathcal{N})$  for any  $a \in \Phi$  satisfying (5.8).

By the way, submodular invariants can usually be found for NIM-reps which lack a true modular invariant. For example, the seemingly extraneous  $n$ -vertex tadpole mentioned in the previous paragraph corresponds to the algebra  $A_1^{(1)}$  at level  $2n - 1$ , and the submodular invariant  $M_{ab} = \delta_{b,J^a a}$ . Perhaps a reasonable interpretation can be found by both the subfactor and boundary CFT camps for NIM-reps corresponding to matrices  $M$  commuting with certain small-index subgroups of  $\text{SL}_2(\mathbb{Z})$ . Recall that we anticipated this thought at the end of Example 1.

Most of the modular invariant classification effort has been directed not at specific algebras and levels, but at the general argument. The major result obtained thus far is:

**THEOREM 5.** [49] *Choose any affine algebra  $X_r^{(1)}$  and level  $k$ . Let  $M$  be any modular invariant, obeying the constraint that the only primaries  $a \in \Phi$  for which  $M_{0a} \neq 0$  or  $M_{a0} \neq 0$ , are units. Then  $M$  lies on an explicit list.*

Note that, of the  $A_1^{(1)}$  modular invariants, all but  $\mathcal{E}_6$  and  $\mathcal{E}_8$  obey the constraint of Thm. 5. That pattern seems to continue for the other algebras and levels: the list of modular invariants covered by Thm. 5 exhausts almost every modular invariant yet discovered.

There are very few *exceptional* modular invariants in the list of Thm. 5. Almost all of the modular invariants there are simple-current ones (5.3), and the product of these by the conjugation  $C$  (strictly speaking, any symmetry of the unextended Dynkin diagram can be used here in place of  $C$ ).

Thm. 5 is important because, *for generic choice of algebra and level*, the various constraints we have on the 0-row and 0-column of a modular invariant (most importantly, Galois (5.4b),  $T$  (5.1), and the inequality (5.5)) force the condition of Thm. 5 to be satisfied.

Indeed, if we impose the full structure of Ocneanu cells [76] (this should be equivalent to saying that an RCFT exists with partition function given by  $M$ ), we obtain Ocneanu's inequality:

$$\sum_{\mu \in \text{clearing}} N_{\lambda, C\lambda}^{\mu} S_{\mu 0} \leq S_{\lambda 0} \quad (6.2)$$

where  $\lambda$  is any weight  $\neq 0$  obeying  $M_{\lambda 0} \neq 0$  with  $\lambda_0$  as large as possible, and where ‘clearing’ is a subset of  $P_+^k$  close to 0:  $\mu$  is in the clearing if  $2(k - \mu_0) \leq k - \lambda_0$ . The left-side of (6.2) grows approximately quadratically with  $S_{\lambda 0}/S_{00}$ , while the right-side is only linear, so it tends to force  $S_{\lambda 0}$  to be small; equation (5.1) on the other hand tends to force  $S_{\lambda 0}$  to be large. This should imply that, for fixed algebra  $X_r^{(1)}$ , there is a  $K$  (depending on the algebra) such that  $\forall k > K$ , the constraint of Thm. 5 will be obeyed! Thus:

**COROLLARY 6.** *All possible modular invariants appearing in RCFT (or the subfactor interpretation), corresponding to any fixed choice of affine algebra  $X_r^{(1)}$ , and all sufficiently high levels, are known.*

In other words, what Cor. 6 tells us is that, apart from some low level exceptional modular invariants, all affine algebra modular invariants appearing in RCFT can be constructed in straightforward and known ways from the symmetries of the corresponding affine Dynkin diagram!

Thm. 5 has another consequence. It makes it relatively easy to find all modular invariants (using only conditions **MI1-MI3**) at ‘small’ levels, when the rank of the algebra isn’t too large [54]. For example, all modular invariants for  $E_8^{(1)}$  at all levels  $k \leq 380$  can be determined. This isn’t completely trivial:  $E_8^{(1)}$  at  $k = 380$  has over  $10^{12}$  highest weights=primaries, so the  $S$  and  $M$  matrices have a number of entries approximately equal to Avogadro’s number! And each of these entries of  $S$ , given by (3.2b), involves a sum of  $10^9$  complex numbers. The fact that we can reach such high levels isn’t a sign of

programming prowess, but rather to how close we are to a complete classification of these (unrestricted) affine algebra modular invariants. In [54] the modular invariants are given for all exceptional algebras, and the classical algebras of rank  $\leq 6$ .

The big surprise here is how rare the affine algebra modular invariants are (for comparison, recall that there are over 8000 modular invariants for the finite group  $A_5$ ). In the Table we've summarised the modular invariant classifications for various algebras of small rank. It describes the complete list of modular invariants for these algebras, when the level is sufficiently small (these limits are given in the Table). A very safe conjecture though is that the Table gives the complete classification for those algebras, for *all* levels  $k$  (at the time of writing,  $E_7^{(1)}$  level 42 and  $E_8^{(1)}$  level 90 still have not been eliminated). Our hope is that this Table (or more realistically, the paper [54] where more results are given and in more detail) will inspire someone to spot a new coincidence involving modular invariants and some other area of mathematics. For example, note in  $A_1^{(1)}$  that the exceptionals appear at  $k+2 = 12, 18, 30$ , which are the Coxeter numbers of  $E_6, E_7, E_8$ . Claude Itzykson noticed that the  $A_2^{(1)}$  exceptionals occur at  $k+3 = 8, 12, 24$  —all divisors of 24— and (inspired by the Fermat connection [79]) found signs of these exceptionals in the Jacobian of  $x^{24} + y^{24} + z^{24} = 0$ . Can anyone spot any such pattern for the other algebras?

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**Table.** Some affine algebra modular invariant classifications

algebra	# of series	levels of exceptionals	verified for:
$A_1^{(1)}$	$k$ odd: 1 $k$ even: 2	$k = 10, 16, 28$	$\forall k$
$A_2^{(1)}$	$k$ arbitrary: 4	$k = 5, 9, 21$	$\forall k$
$C_2^{(1)}$	$k$ arbitrary: 2	$k = 3, 7, 8, 12$	$k \leq 25\,000$
$G_2^{(1)}$	$k$ arbitrary: 1	$k = 3, 4$	$k \leq 30\,000$
$A_3^{(1)}$	$k$ odd: 2 $k$ even: 4	$k = 4, 6, 8$	$k \leq 4000$
$B_3^{(1)}$	$k$ arbitrary: 2	$k = 5, 8, 9$	$k \leq 3000$
$C_3^{(1)}$	$k$ odd: 1 $k$ even: 2	$k = 2, 4, 5$	$k \leq 4500$
$F_4^{(1)}$	$k$ arbitrary: 1	$k = 3, 6, 9$	$k \leq 2000$
$E_6^{(1)}$	$k$ arbitrary: 4	$k = 4, 6, 12$	$k \leq 500$
$E_7^{(1)}$	$k$ odd: 1 $k$ even: 2	$k = 3, 12, 18, (42?)$	$k \leq 400$
$E_8^{(1)}$	$k$ arbitrary: 1	$k = 4, 30, (90?)$	$k \leq 380$